ONLINE TECHNICAL APPENDIX

Proof of Proposition 1

First, we derive the firms’ profits in the subgames for the benchmark case. Table 2 shows the resulting profits, prices and quantities. Below, we derive the profits in the B-B subgame for expositional purposes. The profits in the other subgames are derived in the same manner.

In the B-B subgame, two products are available in the market, since both firms offer the base product only. This implies there exist at most three potential consumer segments. The segment \([x_1, 1]\) buys the highest quality product, which is the base product of firm \(H\), \((m_h)\), the segment \([x_2, x_1]\) buys the lowest quality product \((m_l)\), and the segment \([0, x_2]\) buys nothing. The threshold consumers \(x_1\) and \(x_2\) are indifferent between adjacent quality levels, thus satisfying the equations:

\[
(m_h)x_1 - p_{hb} = (m_l)x_1 - p_{lb} \quad \text{and} \quad (m_l)x_2 - p_{lb} = 0
\]

Hence, \(x_1 = \frac{p_{hb} - p_{lb}}{d}\) and \(x_2 = \frac{p_{lb}}{m_l}\). The quantities sold by the firms are: \(q_{hb} = 1 - x_1\) and \(q_{lb} = x_1 - x_2\); and the objective functions of the firms can be expressed as follows:

\[
\text{Max}_{p_{hb}} \pi_{H}^{B-B} = p_h \left(1 - \frac{p_{hb} - p_{lb}}{d}\right)
\]

\[
\text{Max}_{p_{lb}} \pi_{L}^{B-B} = p_l \left(\frac{p_{hb} - p_{lb}}{d} - \frac{p_{lb}}{m_l}\right)
\]

Differentiating (A.2) with respect to \(p_h\) and (A.3) with respect to \(p_l\) and setting the derivatives equal to zero yields the following solution for equilibrium prices:

\[
p_{hb} = \frac{2d(1+d)}{3+4d} \quad \text{and} \quad p_{lb} = \frac{d}{3+4d}.
\]

(Note that the second order conditions for the maximizations in (A.2) and (A.3) are satisfied because \(\frac{\partial^2 \pi_{H}^{B-B}}{\partial p_{hb}^2} = -\frac{2}{d} < 0\), \(\frac{\partial^2 \pi_{L}^{B-B}}{\partial p_{lb}^2} = -2 - \frac{2}{d} < 0\).)

Substituting the equilibrium prices into the expressions derived for \(x_1\) and \(x_2\), allows us to solve for the quantities sold of each type of product as follows:

\[
q_{hb} = \frac{2(1+d)}{3+4d}, \quad q_{lb} = \frac{1+d}{3+4d}.
\]

Substituting the equilibrium prices and quantities in the profit functions given in (A.2) and (A.3) yields the equilibrium profits, \(\pi_{H}^{B-B} = \frac{4d(1+d)^2}{(3+4d)^2}\) and \(\pi_{L}^{B-B} = \frac{4d(1+d)^2}{(3+4d)^2}\). Similarly, we derive the profits of firm \(H\) and \(L\) in all other subgames.
The assumptions to ensure a positive amount of sales for all products in the benchmark equilibrium are as follows. In the \( B-F \) equilibrium, \( c < \frac{2(1+\alpha)(1+(1+\alpha)d)}{1+2(1+\alpha)d} \); in the \( F-B \) equilibrium, \( c < \frac{(1-\alpha)(1+\alpha)d}{1+(2-\alpha)d} \); and in the \( F-F \) equilibrium, \( c < \frac{1+\alpha d}{2} \).

We now establish the equilibrium conditions for the strategy combinations.

\textbf{B-B}

We examine firms’ incentive to deviate from the \( B-B \) subgame. First, considering firm H’s deviation from \( B-B \) to \( B-F \) by examining \( \pi_{H}^{B-B} - \pi_{H}^{B-F} \),
\[
\pi_{H}^{B-B} - \pi_{H}^{B-F} = \frac{1}{144} [72c - 36\alpha d - \frac{27}{(3+4d)^2} - \frac{45}{3+4d} - \frac{16c^2}{(3+4(1+\alpha)d)^2} + \frac{3(3-4c)^2}{3+4(1+\alpha)d} + \frac{45-80c^2}{3+4(1+\alpha)d}] \tag{A.6}
\]

The right-hand side of (A.6) is positive for \( c > \lambda_{3}^{BM} \), where \( \lambda_{3}^{BM} = \frac{1}{2} + (1 + \alpha)d - \frac{1}{2+4(1+\alpha)d} \). Considering firm L’s deviation, we have
\[
\pi_{L}^{B-B} - \pi_{L}^{B-F} = \frac{d(1+d)(3+4(1+\alpha)d)}{(3+4d)^2} \left[ \frac{(1+d)(c+2(1-\alpha)d-(1-\alpha)d(1+\alpha)d)^2}{(1-\alpha)d(3+4-\alpha)d^2(1+\alpha)d} \right] \tag{A.7}
\]

The right-hand side of (A.7) is positive for \( c > \lambda_{4}^{BM} \), where \( \lambda_{4}^{BM} = \frac{(1-\alpha)(1+\alpha)d}{1+(2-\alpha)d} \). Note that \( \lambda_{4}^{BM} < \lambda_{3}^{BM} \). Thus, \( B-B \) is an equilibrium when \( c > \lambda_{3}^{BM} \).\(^{10}\)

\textbf{B-F}

Similar to above, we consider firms’ potential deviation from \( B-F \) equilibrium. First, (A.6) shows that firm \( H \) chooses \( B-F \) over \( B-B \) for \( c < \lambda_{3}^{BM} \). Considering firm \( L \) deviation from \( B-F \) to \( F-F \),
\[
\pi_{L}^{B-F} - \pi_{L}^{B-F} = \frac{1+(1+\alpha)d}{d} \left[ \frac{(c+(1+\alpha)d)^2}{(1+\alpha)(3+4(1+\alpha)d)^2} - \frac{d^2(1+\alpha d-2c)^2}{(1+\alpha d)(3+4+3\alpha d)^2} \right] \tag{A.8}
\]

The right-hand side of (A.8) is positive if \( c > \lambda_{2}^{BM} \), where
\[
\lambda_{2}^{BM} = \frac{(1+\alpha)(1+\alpha)[9+(42+18\alpha+64\alpha+9(8+\alpha)d+32(1+\alpha)^2d^2)]d-3(3+4(1+\alpha)d)(1+2+3\alpha)d(3+(4+3\alpha)d)d\sqrt{(1+\alpha)(1+\alpha)}[64(1+\alpha)^3d^2+(96+176+9(8-\alpha)\alpha)d^2-4(6-5d)-27\alpha-3\alpha(4+9\alpha)d]d-9}{64(1+\alpha)^3d^2+(96+176+9(8-\alpha)\alpha)d^2-4(6-5d)-27\alpha-3\alpha(4+9\alpha)d]d-9}
\]

Further, the technical condition to guarantee \( q_{hf} > 0 \) is \( c < \frac{2(1+\alpha)(1+(1+\alpha)d)}{1+2(1+\alpha)d} \). Comparing it with

\(^{10}\)Note that the technical conditions for positive sales are always satisfied.
\( \chi_3^{BM} \), we have \( \chi_3^{BM} - \frac{2(1+\alpha)(1+\alpha)d}{1+2(1+\alpha)d} = -2 \sqrt{\frac{(1+\alpha)(1+d)^2(3+4+4(1+\alpha)d)^2d^2}{(3+4d^2(1+2(1+\alpha)d)^2}} < 0 \). Thus, \( B-F \) is an equilibrium when \( \chi_2^{BM} < c < \chi_3^{BM} \).

**F-B**

(A.7) shows that firm \( L \) chooses \( F-B \) over \( B-B \) for \( c < \chi_4^{BM} \). Considering firm \( H \)'s potential deviation from \( F-B \) to \( F-F \),

\[
\pi_H^{F-B} - \pi_H^{F-F} = \frac{4(2(1-\alpha)+c)^2}{(7-\alpha)^2(1-\alpha)} - \frac{(2(2+\alpha)-c)^2}{(7+3\alpha)^2} \tag{A.9}
\]

The right-hand side of (A.9) is positive when \( c > \chi_1^{BM} \), where

\[
\chi_1^{BM} = 2d\left[\frac{1}{9+3(2+d)23a^2d(3+4d)+3(4+3d)(2+5+4d)d\alpha+42+a^3d^2d)} - \frac{1}{9+3(2+d)23a^2d(3+4d)+3(4+3d)(2+5+4d)d\alpha+42+a^3d^2d)}\right].
\]

Note that \( \chi_1^{BM} > \chi_4^{BM} \). Thus, \( F-B \) cannot arise as equilibrium.

**F-F**

(A.9) shows that firm \( H \) chooses \( F-F \) over \( F-B \) for \( c < \chi_1^{BM} \), and (A.8) shows that firm \( L \) chooses \( F-F \) over \( B-F \) for \( c < \chi_2^{BM} \). The technical condition to guarantee positive sales for both products is \( c < \frac{1+\alpha d}{2} \). Comparing it with \( \chi_1^{BM} \) and \( \chi_2^{BM} \), we have \( \frac{1+\alpha d}{2} > \chi_1^{BM} \) and \( \frac{1+\alpha d}{2} > \chi_2^{BM} \).

Therefore, \( F-F \) is an equilibrium for \( c < \min(\chi_1^{BM}, \chi_2^{BM}) \). To show the existence of the equilibria in Proposition 1, consider an example with \( d = 1 \) and \( \alpha = 0.5 \). Then, we have \( \chi_1^{BM} = 0.2769, \chi_2^{BM} = 0.0283 \) and \( \chi_3^{BM} = 0.3008 \), showing existence of the equilibria for appropriate values of \( c \). For this example, the technical condition required for positive sales of both firms in all equilibria is \( c < 0.75 \), which holds because \( c < \alpha = 0.5 \), by assumption.

**Proof of Lemma 1**

First, we derive the profits in the nine subgames listed in Table 1. We present below the analysis of the \( B-BF \) subgame. The derivations for the other subgames are similar and are omitted.

In \( B-BF \), three products are available in the market; firm \( L \) offers a base product only, and firm \( H \) offers both a base product and a feature-added one. Thus, there exist at most four consumer segments. The segment \([x_0, 1]\) buys the highest quality product, which is the product with feature of firm \( H (m_h + ad) \), the segment \([x_1, x_0)\) buys the intermediate quality product, the base product of firm \( H (m_h) \), the segment \([x_2, x_1)\) buys the lowest quality product, the base product of firm \( L \)
(m_h) and the segment $[0,x_2)$ nothing. The threshold consumers, $x_0$, $x_1$ and $x_2$, are indifferent between adjacent quality levels, thus satisfying the equations:

$$(m_h + ad)x_1 - p_{hf} = (m_h)x_1 - p_{hb} \quad , \quad (m_h)x_2 - p_{hb} = (m_l)x_2 - p_{lb} \quad \text{and} \quad (m_l)x_3 - p_{lb} = 0 \quad (A.10)$$

Solving, we get $x_0 = \frac{p_{hf} - p_{hb}}{ad}$, $x_1 = \frac{p_{hb} - p_{lb}}{d}$, and $x_2 = \frac{p_{lb}}{m_l}$. The quantities sold by the firms are: $q_{hf} = 1 - x_0$, $q_{hb} = x_0 - x_1$ and $q_{lb} = x_1 - x_2$; and the objective functions of the firms can be expressed as follows:

$$\max_{p_{hf}, p_{hb}} \pi^{B-BF}_H = (p_{hf} - c) \left(1 - \frac{p_{hf} - p_{hb}}{ad}\right) + (p_{hb}) \left(\frac{p_{hf} - p_{hb}}{ad} - \frac{p_{hb} - p_{lb}}{d}\right), \quad (A.11)$$

$$\max_{p_{lb}} \pi^{B-BF}_L = (p_{lb}) \left(\frac{p_{hb} - p_{lb}}{d} - \frac{p_{lb}}{m_l}\right). \quad (A.12)$$

Differentiating (A.11) with respect to $p_{hf}$, $p_{hb}$ and (A.12) with respect to $p_{lb}$ and setting the derivatives equal to zero, yields the following solution for equilibrium prices:

$$p_{hf} = \frac{c(3 + 4d) + d(4 + 3a + 4(1+a)d)}{6 + 8d}, p_{hb} = \frac{2d(1+d)}{3 + 4d}, \text{ and } p_{lb} = \frac{d}{3 + 4d} \quad (A.13)$$

Note that the second order conditions for the maximizations in (A.11) and (A.12) are satisfied because 

$$\frac{\partial^2 \pi^{B-BF}_H}{\partial p_{hf}^2} = -\frac{2}{ad} < 0, \quad \frac{\partial^2 \pi^{B-BF}_H}{\partial p_{hb}^2} = -\frac{2}{ad} < 0, \quad \frac{\partial^2 \pi^{B-BF}_L}{\partial p_{lb}^2} = -2 - \frac{2}{d} < 0$$

Substituting the equilibrium prices into the expressions derived for $x_1$, $x_2$ and $x_3$ allows us to solve for the quantities sold of each type of product as follows:

$$q_{hf} = \frac{1}{2} - \frac{c}{2ad}, \quad q_{hb} = \frac{3c + ad + 4cd}{6ad + 8ad^2}, \quad q_{lb} = \frac{1 + d}{3 + 4d} \quad (A.14)$$

Substituting the equilibrium prices and quantities back into the objectives (A.11) and (A.12) yields the equilibrium profits, $\pi^{B-BF}_H = \frac{1}{16} \frac{4c^2}{ad} + \frac{4d(16(1+d)^2 + a(3+4d)^2)^2}{(3+4d)^2} - 8c$ and $\pi^{B-BF}_L = \frac{d(1+d)}{(3+4d)^2}$.

Table 3 summarizes the equilibrium profits, prices and quantities for all subgames. We now determine which subgames fail to become an equilibrium by comparing the profits. First, for the $B$-$B$ subgame, we examine if firm $H$ has any incentive to deviate from its strategy.

$$\pi^{B-B}_H - \pi^{B-BF}_H = -\frac{(c-ad)^2}{4ad} \quad (A.15)$$
The right-hand side of (A.15) is negative. Thus, B-B fails to become equilibrium. Likewise, for the subgames BF-B and F-B, we examine firm H’s potential deviation.

\[
\pi_H^{BF-B} - \pi_H^{BF-BF} = -\frac{(c-ad)^2}{4ad} < 0 \tag{A.16}
\]

\[
\pi_H^{B-B} - \pi_H^{B-BF} = -\frac{(c-ad)^2}{4ad} < 0 \tag{A.17}
\]

(A.16) and (A.17) show that both BF-F and F-B cannot become equilibrium. For the subgames F-BF and F-F, we examine firm L’s potential deviation.

\[
\pi_L^{F-BF} - \pi_L^{BF-BF} = -\frac{c^2}{4ad+4\alpha^2d^2} < 0 \tag{A.18}
\]

\[
\pi_L^{F-F} - \pi_L^{BF-F} = -\frac{c^2}{4\alpha d+4\alpha^2 d^2} < 0 \tag{A.19}
\]

Thus, F-BF and F-F cannot arise as equilibrium.

**Proof of Proposition 2**

Given the equilibrium profits in Table 3, we establish the equilibrium regions for the assuming \(d = 1\).

**B-BF**

By examining firm H’s deviation between B-BF and B-F, we have the following:

\[
\pi_H^{B-BF} - \pi_H^{B-BF} = \frac{16}{49} + \frac{\alpha}{4} - \frac{(4+2(3+\alpha-c)(\alpha-3\alpha)^2}{(1+\alpha)(7+4\alpha)^2} - \frac{c}{2} + \frac{c^2}{4\alpha} \tag{A.20}
\]

The right-hand side of (A.20) is positive if \(c > \chi_2 = \frac{1}{7}[2\sqrt{3} \sqrt{\frac{\alpha^2(1+\alpha)^2(7+4\alpha)^2(24+19\alpha)}{49+3\alpha(23+8\alpha)^2}} + \frac{7\alpha(1+\alpha)}{49+3\alpha(23+8\alpha)}]\).

Now, we examine if firm L has any incentive to deviate to BF-BF from B-BF when \(c > \chi_2\). By comparing firm L’s profits in B-BF and BF-BF, we have the following:

\[
\pi_L^{B-BF} - \pi_L^{BF-BF} = \frac{4(25\alpha-7)(\alpha+196(3-\alpha)c}{49(7-\alpha)^2} - \frac{(49-\alpha(40-7\alpha)c^2}{4\alpha(7-\alpha)^2(1-\alpha)} \tag{A.21}
\]

The right-hand side of (A.21) is positive for \(c > \chi_3\), where \(\chi_3 = \frac{4}{7}\sqrt{\frac{14(3-\alpha)(1-\alpha)\alpha}{49-(40-7\alpha)^2} - \sqrt{\frac{(7-\alpha)^2(1-\alpha)(29-21\alpha)}{49-(40-7\alpha)^2}}]}\). Note that \(\chi_2 > \chi_3\) for \(0 < \alpha < d\). Moreover, the technical condition to guarantee \(q_{hf} > 0\) is \(c < \alpha\). Thus, B-BF is an equilibrium when \(c > \chi_2\).
B-F

(A.20) shows that firm $H$ chooses $B-F$ over $B-BF$ if $c < \chi_2$. For firm $L$, examining

$$\pi_L^{B-F} - \pi_L^{B-F-F}$$
gives

$$\pi_L^{B-F} - \pi_L^{B-F-F} = \frac{(2+\alpha)(1+\alpha+c)^2}{(1+\alpha)(7+4\alpha)^2} - \frac{4\alpha(1+\alpha)(2+\alpha)-16\alpha(2+\alpha)c+(49+25\alpha)c^2}{4\alpha(7+3\alpha)^2}$$

(A.22)

The right-hand side of (A.22) is positive if $c > \chi_1$, where

$$\chi_1 = \frac{2(1+\alpha)(2+\alpha)}{1+2(1+\alpha)}$$

The technical condition for $q_{hf} > 0$ is

$$c < \frac{2(1+\alpha)(2+\alpha)}{1+2(1+\alpha)}$$

Thus, $B-F$ is an equilibrium if $\chi_1 < c < \chi_2$.

BF-F

(A.22) shows that firm $L$ chooses $BF-F$ over $B-F$ if $c < \chi_1$. For firm $H$, we examine whether firm $H$ has any incentive to deviate from $BF-F$ to $BF-BF$ if $c < \chi_1$.

$$\pi_H^{BF-F} - \pi_H^{BF-BF} = \frac{(4+2\alpha-c)^2}{(7+3\alpha)^2} + \frac{2(17-(14-\alpha)\alpha)c-64+(15-\alpha)(1+\alpha)a}{4(7-\alpha)^2} - \frac{[49-(47-(15-\alpha)\alpha)a]c^2}{(4(7-\alpha)^2(1-\alpha)a)}$$

(A.23)

The right-hand side of (A.23) is positive for $c < \chi_1$. The technical condition to ensure

$$q_{hf}, q_{lf} > 0$$
is

$$c < \frac{2(2+\alpha)}{7+4\alpha} \cdot \frac{2(2+\alpha)}{7+4\alpha} > \chi_1$$

because $\chi_1 - \frac{2(2+\alpha)}{7+4\alpha} > 0$ for $0 < \alpha < 1$. Thus $BF-F$ is an equilibrium if $c < \chi_1$.

BF-BF

(A.23) shows that firm $H$ chooses $BF-BF$ over $BF-F$ if $c > \chi_1$, and (A.21) shows that firm $L$ chooses $BF-BF$ over $B-BF$ if $c < \chi_3$. However, $\chi_3 < \chi_1$ if $0 < \alpha < d$. Thus, $BF-BF$ cannot arise as equilibrium.

To show the existence of the equilibrium strategies, consider the example where $\alpha = 0.5$. Then, we have $\chi_1 = 0.0298$ and $\chi_2 = 0.0184$, showing that all equilibria can exist for appropriate values of $c$. The technical conditions are $c < 0.278$ for $BF-F$, $c < 1.875$ for $B-F$, and $c < 4$ for $B-BF$, which are all satisfied.

Proof of Result 1
The right-hand side of (A.24) is positive.

\[
\frac{\partial x_2}{\partial \alpha} = \frac{(7+5\alpha)(7+9\alpha)}{(49+3(23+8\alpha)\alpha)^2} + \frac{3\sqrt{3}(5488+(21021+(31387+2(11513+8(523+76\alpha)\alpha)\alpha)\alpha)}

A.24

The right-hand side of (A.25) is positive for \(0 < \alpha < 1\).

\[
\frac{\partial x_1}{\partial \alpha} = \frac{196(6-\sqrt{29})+4(1498-245\sqrt{29}+(2073-330\sqrt{29}+(1108-170\sqrt{29}+5(41-6\sqrt{29})\alpha)\alpha)\alpha)}{2401+(5978+(5621+4(586+91\alpha)\alpha)\alpha)\alpha}

A.25

\]

The right-hand side of (A.27) is positive, implying \(x_1 > \min(x_1^{BM}, x_2^{BM})\).

\[
\pi_L^{BF-F} - \pi_L^{F-F} (BM) = \frac{c^2}{4\alpha(1+\alpha)}

A.28

The right-hand side of (A.28) is positive, meaning firm L’s profit is higher in the BF-F equilibrium of the main analysis. Now considering firm H’s profit, Tables 2 and 3 show that \(\pi_H^{BF-F} = \pi_H^{F-F} (BM)\).

Second, when \(\min(x_1^{BM}, x_2^{BM}) < c < x_1\), we compare firms’ profit in the BF-F equilibrium with that in the B-F benchmark equilibrium. For firm L,

\[
\pi_L^{BF-F} - \pi_L^{F-F} (BM) = \frac{(49+25\alpha)c^2+4(1+\alpha)(2+\alpha)\alpha-16(2+\alpha)\alpha c}{4\alpha(7+3\alpha)^2} - \frac{(2+\alpha)(1+\alpha+c)^2}{(1+\alpha)(7+4\alpha)^2}

A.29

The right-hand side of (A.29) is positive for \(c < x_1\), showing firm L has a higher profit in the BF-F equilibrium of the main analysis. Now, considering firm H’s profit, we examine \(\pi_H^{BF-F} - \pi_H^{F-F}\).
The right-hand side of (A.30) is negative. Thus, firm H’s profit in the BF-F equilibrium is lower than the profit in the B-F benchmark equilibrium.

We now consider the case when firm H expands its product line (i.e., when \( c \) is sufficiently large). First, when \( \chi_{3}^{BM} < c \), we compare firm H’s profit in the B-BF equilibrium of the main analysis with that in the B-B benchmark equilibrium.

\[
\pi_{H}^{B-BF} - \pi_{H}^{B-B} (BM) = \frac{(a-c)^2}{4a} \tag{A.31}
\]

The right-hand side of (A.31) is positive, meaning firm H’s profit is higher in the B-BF equilibrium of the main analysis. Now, considering firm L’s profit, Table 1 and 2 show \( \pi_{L}^{B-BF} = \pi_{L}^{B-B} (BM) \).

Second, when \( \chi_{2} < c < \chi_{3}^{BM} \), comparing firm H’s profit with that in the B-F benchmark equilibrium,

\[
\pi_{H}^{B-BF} - \pi_{H}^{B-F} (BM) = \frac{16}{49} + \frac{a}{4} + \frac{c^2}{4a} - \frac{[4+2(3+a-c)a-3c]^2}{(1+a)(7+4a)^2} - \frac{c}{2} \tag{A.32}
\]

The right-hand side of (A.32) is positive for \( c \geq \chi_{2} \), showing firm H has a higher profit in the B-BF equilibrium of the main analysis. Considering firm L’s profit,

\[
\pi_{L}^{B-BF} - \pi_{L}^{B-F} (BM) = \frac{d(1+d)}{(3+4d)^2} - \frac{(1+d+ad)(c+d+ad)^2}{(1+a)d(3+4(1+a)d)^2} \tag{A.33}
\]

The right-hand side of (A.33) is negative. Thus, firm L cannot receive higher profit in the B-BF equilibrium than the B-B and B-F benchmark equilibria.