Math Meth Oper Res
DOI 10.1007/s00186-012-0407-4

ORIGINAL ARTICLE

Tackling indeterminacy in overlapping generations models

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Abstract Overlapping generations models may have a continuum of equilibria. Previous studies have been largely confined to the local analysis that linearizes the model around the steady state. However, what is true of the linearized system only applies for an unknown-sized open neighborhood of the steady state. In this paper I develop a method to diagnose the indeterminacy in overlapping generations models by computing the set of all equilibria. I also provide a procedure to simulate the economy with indeterminate equilibrium.

Keywords OLG · Indeterminacy · Computation · Simulation

1 Introduction

This paper studies how to diagnose indeterminacy of equilibrium in overlapping generations (OLG) models and how to simulate the economy when there is indeterminacy of equilibrium. It is well known that an overlapping generations economy may have a continuum of equilibria. There are well-established results that provide sufficient conditions for the uniqueness of the equilibrium in these economies. Gale (1973) has demonstrated that gross substitution in consumption precludes indeterminacy in OLG models with one good in each period and a single two-period-lived consumer in each generation. This result has been extended to a multi-commodity economy where a single two-period-lived consumer is characterized by log-linear preference (Balasko and Shell 1980) and inter-temporally separable preferences (Geanakoplos and Polemarchakis 1984; Kehoe and Levine 1984). Kehoe et al. (1991) further extend

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Published online: 09 September 2012

these results to a multi-commodity, multi-agent, non-monetary, pure-exchange economy and show that gross substitutability of excess demand ensures the determinacy of perfect-foresight equilibria. Unfortunately, the condition for uniqueness in the model of the type that has been widely used in the macro literature, like the model discussed in this paper, is not guaranteed based on the empirical evidences provided in Mankiw et al. (1985).

We can identify the existence of indeterminacy by linearizing the model around the steady state when the model is deterministic. However, what is true of the linearized system only applies for an open neighborhood of the steady state. In practice, we do not know the size of this open neighborhood as pointed out by Kehoe and Levine (1990). Furthermore, we can not say anything outside of this neighborhood based on this linearized system. The existence of indeterminate equilibria still poses challenge for the economists who are interested in conducting comparative statics analysis, since there exists no simulation method for this type of economy.

In this paper, I provide a global analysis by computing all Markovian equilibria using the numerical procedure set forth in Feng et al. (2011). Therefore, my approach provides a straightforward way to diagnose the indeterminacy by simply checking the numerically obtained set of equilibria. The analysis departs from the economy considered by Kehoe and Levine (1990), in which we know that the economy has indeterminate equilibria around one steady state. I verify the existence of indeterminacy by computing the set of all Markov equilibria.

It has been theoretized that a continuum of equilibria will all converge to the same steady state asymptotically in a deterministic OLG model with indeterminate equilibrium (see Spear et al. 1990; Wang 1993). To my best knowledge, this has not been verified due to the lack of a robust algorithm to compute the equilibrium set of these models. I simulate these economies by computing all Markov equilibrium in the spirit of Feng et al. (2011). I then study whether or not and how the existence of indeterminacy may affect the long run behavior of the economy. Numerical simulations indicate that in a deterministic OLG model considered by this study, all equilibrium paths asymptotically converge to the same steady state. However the choice of initial conditions will translate into different equilibrium paths before they converge to the long-run equilibrium.

I proceed as follows. In Sect. 2, I explain the economic model. Section 3 present a numerical method to identify indeterminacy in these economies. I also consider two examples to illustrate the application. In Sect. 4, I explain how to simulate the OLG models with indeterminate equilibria. I also discuss how the indeterminacy may affect the long run behavior of the economy. Some further comments and extensions follow in the final section.

2 Model

The economy is conformed by a sequence of overlapping generations. Time is discrete. At every time period $t = 0, 1, 2, \ldots$, a new generation is born. Each generation is made up of a representative agent, who is present in the economy for three periods. The individual is identified by the date of her birth, t, and her age $a = 0, 1, 2$. The agent born at time t consumes and has endowments at all time periods $t+a, a=0,1,2$. Let c_t^a and e_t^a denote her consumption and endowment when her age is a. An agent's individual endowments only depend on her age, i.e. for all $a=0,1,2, e_t^a=\mathbf{e}^a$.

At each period t, there exist spot markets for the consumption good. There is one bond in zero net supply. The bond pays one unit consumption good next period. Price of the bond is $q_t \in \mathbf{R}$ and agent a's bond holding is $\theta_t^a \in \mathbf{R}$. At $t=0$, the initial conditions of the economy are determined by the bond holding of the initially alive agents of ages $a=1,2$.

For simplicity, I assume that each agent's utility function U is separable over consumption of different dates. The inter-temporal objective U is defined as

$$U(c)=\sum_{a=0}^{2}\beta^a u(c_t^a) \tag{1}$$

The one-period utility u satisfies the following conditions:

Assumption 2.1 The one-period utility function $u(\cdot):\mathbf{R}_+\rightarrow\mathbf{R}\cup\{-\infty\}$ is increasing, strictly concave, and continuous. This function is also continuously differentiable at every interior point $c>0$.

Definition 1 A sequential competitive equilibrium (SCE) is given by a collection of prices and choices of individuals $\{q_t,(c_t^a,\theta_t^a)_{a=0,1,2}\}_t$ such that for each t :

(i) financial market clearing:

$$\sum_{a=0}^{2}\theta_t^a=0 \tag{2}$$

(ii) individual maximizes utility:

$$(c_t^a,\theta_t^a)\in\arg\max U(c),\quad s.t. \tag{3}$$
$$c_t^0+q_t\cdot\theta_t^0\leq e_t^0 \tag{4}$$
$$c_t^1+q_{t+1}\cdot\theta_t^1\leq e_t^1+\theta_t^0 \tag{5}$$
$$c_t^2\leq e_t^2+\theta_t^1 \tag{6}$$

The existence of a SCE can be established by standard methods (e.g., Balasko and Shell 1980; Schmachtenberg 1998). Moreover, by similar arguments used by these authors it is easy to show that every sequence of equilibrium asset prices $(q_t)_{t\geq 0}$ is bounded.

2.1 Indeterminacy in OLG models

It is well known that an overlapping generations economy may have a continuum of equilibria. Unfortunately, the existing studies have been largely confined to the local analysis that linearizes the model around the steady state. More specifically, economists linearize the equilibrium conditions around a steady state and then solve the linearized version of the model.

It is convenient to build market-clearing into the first order conditions and the steady state can be characterized by the solutions of the following equations.

$$u_c(e_t^0 + q_t\theta_{t-1}) - \beta u_c(e_t^1 - \theta_t - q_{t+1}\theta_{t+1}) = 0 \tag{7}$$
$$u_c(e_t^1 - \theta_t - q_{t+1}\theta_{t+1}) - \beta u_c(e_t^2 + \theta_{t+1}) = 0 \tag{8}$$
$$q_t = q_{t+1} = q^* \tag{9}$$
$$\theta_{t-1} = \theta_t = \theta_{t+1} = \theta^* \tag{10}$$

where I have build in the bond market clearing condition.

Evidently, the system always has a solution with $q^* = 1$, corresponding to the golden rule monetary steady state. However, it is more interesting to check how many other solutions, referred to as real steady states in Kehoe and Levine (1990), exist in the economy. Following Kubler and Schmedders (2010), I isolate the variable q and characterize all real steady states as the positive real solution ($q^* \neq 1$) to the following equation,

$$f(q) = 0. \tag{11}$$

It is convenient to rewrite the equilibrium conditions (7) and (8) as

$$F(\theta_{t-1}, \theta_t, \theta_{t+1}) = 0 \tag{12}$$

Then I linearize the system as

$$D_1F\theta_{t-1} + D_2F\theta_t + D_3F\theta_{t+1} = 0 \tag{13}$$

Here D_iF is the derivative of F evaluated at the steady state (q^*, θ^*). Rewriting this linearized equilibrium condition as a first-order difference equation, I obtain:

$$\begin{bmatrix} \theta_{t+1} \\ \theta_t \end{bmatrix} = \begin{bmatrix} -D_1F_1 \cdot D_2F_1^{-1} & 0 \\ 0 & -D_1F_2 \cdot D_2F_2^{-1} \end{bmatrix} \begin{bmatrix} \theta_t \\ \theta_{t-1} \end{bmatrix} \tag{14}$$

Indeterminacy of the linearized system manifests itself as too many stable eigenvalues of the matrix in (14). The advantage of this approach is that indeterminacy of the linearized system is easy to diagnose. However, this approach cannot provide any information about the economy away from the steady state since it is only valid for an unknown-sized open neighborhood around the steady state. One task of this paper is to provide a general approach to identify the existence of indeterminacy for OLG models. As I describe the approach in the following section, I will also explain a global analysis for the impact of indeterminacy on the long-run behavior of the economy.

3 Diagnose of indeterminacy

Feng et al. (2011) develop a method to approximate the set of all Markovian equilibria for dynamic equilibrium models. Their study suggests that one can identify the existence of continuous Markov equilibrium by computing the boundary of the set of all Markov equilibria. The numerical example in section (6.3) of their paper shows that the distance between the upper boundary and the lower one will go to zero whenever the equilibrium is unique. This result has important implications for diagnosing the indeterminacy in OLG models.

In this section, I first briefly explain the main results in their paper. I then detail the procedure of computing the correspondence that contains the set of all Markov equilibria. Based on this approximation, two propositions will be derived to identify the indeterminacy.

3.1 Markov equilibrium correspondence

In the model economy, the equilibrium equations consist of first order conditions, budget constraints, and market-clearing conditions. It is useful to build market-clearing into the endogenous choice of security θ^0, θ^1. Then the natural state space Θ consists of beginning-of-period bond-holding of the middle aged θ. Let m denote the consumption of the middle-aged[1]

$$m = c^1. \tag{15}$$

The Markov equilibrium correspondence $\mathbf{V}^* : \Theta \to \mathbf{R}$ is defined as

$$\mathbf{V}^* (\theta_0) = \left\{ m : \left(q, (c^a, \theta^a)_{a=0,1,2} \right) \text{ is a SCE} \right\}. \tag{16}$$

The above results on the existence of SCE for OLG economies leads to the following proposition.

Proposition 1 *Correspondence* $\mathbf{V}^*$ *is nonempty, compact-valued, and upper semi-continuous.*

From this correspondence $\mathbf{V}^*$, I can generate recursively the set of equilibria since $\mathbf{V}^*$ is the fixed point of an operator $\mathbf{B} : \mathbf{V} \to \mathbf{B}(\mathbf{V})$ that links state variables to future equilibrium states. This operator embodies all equilibrium conditions. More precisely, let $\mathbf{B}(\mathbf{V})(\theta)$ be the set of all values m with the following property: for any given θ there exists $\theta_+ = g^m(\theta, m)$ and $m_+(\theta_+) \in \mathbf{V}(\theta_+)$ such that

$$q \cdot u_c(c^0) = \beta u_c \left(m_+(\theta_+) \right) \tag{17}$$

[1] In their original paper, m is defined as the vector of shadow values of the marginal return to investment for all assets: $m = qu'(c^1)$. Here I define m as c for the purpose of computation only. It turns out to be equivalent in all examples I considered in this paper.

and market clearing conditions.[2] The following result is proved in Feng et al. (2011).

Theorem 2 (convergence) *Let* $\mathbf{V}_0$ *be a compact-valued correspondence such that* $\mathbf{V}_0 \supset \mathbf{V}^*$. *Let* $\mathbf{V}_n = \mathbf{B}(\mathbf{V}_{n-1}), n \geq 1$. *Then,* $\mathbf{V}_n \to \mathbf{V}^*$ *as* $n \to \infty$. *Moreover,* $\mathbf{V}^*$ *is the largest fixed point of the operator* $\mathbf{B}$, *i.e., if* $\mathbf{V} = \mathbf{B}(\mathbf{V})$, *then* $\mathbf{V} \subset \mathbf{V}^*$.

3.2 Numerical approximation of $\mathbf{V}^*$

The numerical implementation of the operator $\mathbf{B}$ consists of two parts. In the first step, I construct an operator $\mathbf{B}^{\mathbf{h},\mu}$ and obtain a convex-valued correspondence $\tilde{\mathbf{V}}^*$ containing the equilibrium set $\mathbf{V}^*$. Then, a discretized operator $\mathbf{B}^{h,\mu,N}$ will be developed to approximate $\mathbf{V}^*$ in the next section.

The algorithm starts with an initial correspondence $\tilde{\mathbf{V}}_0 \supseteq \mathbf{V}^*$. I pick an arbitrary small positive number h, μ, and I partition the state space Θ into N closed intervals Θ^i of uniform length h such that $\cup_i \Theta^i = \Theta$ and $int(\Theta^{i_1}) \cap int(\Theta^{i_2}) = \emptyset$ for every pair $\Theta^{i_1}, \Theta^{i_2}$, where $i_1, i_2 \in \{1, 2, ..., N\}$. I use N right rectangular parallelepiped $\tilde{\mathbf{V}}_0^i :=$ $\Theta^i \times \left[\inf_{\theta \in \Theta^i} \tilde{\mathbf{V}}_0(\theta), \sup_{\theta \in \Theta^i} \tilde{\mathbf{V}}_0(\theta)\right]$ to approximate $\tilde{\mathbf{V}}_0 = \cup_i \tilde{\mathbf{V}}_0^i$. It turns out to be convenient to characterize $\tilde{\mathbf{V}}_0^i$ by two functions $\mathbf{m}_0^{\sup}(\theta) = \sup_{\theta \in \Theta^i} \tilde{\mathbf{V}}_0(\theta)$, $\mathbf{m}_0^{\inf}(\theta) = \inf_{\theta \in \Theta^i} \tilde{\mathbf{V}}_0(\theta)$. Then $\tilde{\mathbf{V}}_0^i$ is defined as $\tilde{\mathbf{V}}_0^i := \left\{m | \theta \in \Theta^i, m \in \left[\mathbf{m}_0^{\inf}(\theta), \mathbf{m}_0^{\sup}(\theta)\right]\right\}$.

Consider then any element Θ^i of the state space partition and the corresponding $\tilde{\mathbf{V}}_0^j$. Given $\theta \in \Theta^i$, I test whether there exists $m \in \left[\mathbf{m}_0^{\inf}(\theta), \mathbf{m}_0^{\inf}(\theta) + \mu\right]$ such that the one period temporary equilibrium conditions can be satisfied for some arbitrary small constant $\epsilon > 0$. If the answer is yes, then I set $\mathbf{m}_1^{\inf}(\theta) = \mathbf{m}_0^{\inf}(\theta)$, otherwise, I set $\mathbf{m}_1^{\inf}(\theta) = \mathbf{m}_0^{\inf}(\theta) + \mu$. A symmetric operation is performed for the case $\theta \in \Theta^i$, $m \in \left[\mathbf{m}_0^{\sup}(\theta) - \mu, \mathbf{m}_0^{\sup}(\theta)\right]$. At the end of this operation, an approximation of $\mathbf{V}^*$ will be given by $\tilde{\mathbf{V}}^* = \cup_i \tilde{\mathbf{V}}^*_{\theta \in \Theta^i}(\theta)$. The details of the algorithm can be found in the appendix.

3.3 Sufficient condition for determinacy in OLG economy

A straightforward application of Theorem 2 is that we cannot rule out the possibility of indeterminacy if the above procedure converges to a convex-valued set $\tilde{\mathbf{V}}^*$ that the distance between the upper boundary and the lower one is strictly greater than zero.

Proposition 2 (indeterminacy) *If there is indeterminacy in the model economy, then there exists* $\delta \geq \mu$ *such that* $\max_{(\theta,s) \in \Theta \times \mathbf{S}} \left\{\mathbf{m}^{\sup^*}(\theta, s) - \mathbf{m}^{\inf^*}(\theta, s)\right\} > \delta$ *as* $h \to 0$, $\mu \to 0$.

[2] For any given θ and $m \in \mathbf{V}$, θ_+ is determined as the solution to the following equations

$$m = e^1 + \theta - q\theta_+ \tag{18}$$

$$q \cdot u_c(m) = \beta u_c(e^2 + \theta_+). \tag{19}$$

Proposition 3 (determinacy) *There is no indeterminacy in the model economy if for any arbitrary* $\delta > 0$, $\max_{(\theta,s)\in\Theta\times\mathbf{S}} \left\{\mathbf{m}^{\sup^*}(\theta, s) - \mathbf{m}^{\inf^*}(\theta, s)\right\} \leq \delta$ *as* $h \to 0$, $\mu \to 0$.

Proof *In the limit, we have* $\max_{(\theta,s)\in\Theta\times\mathbf{S}} \left\{\mathbf{m}^{\sup^*}(\theta, s) - \mathbf{m}^{\inf^*}(\theta, s)\right\} = 0$. *This immediately implies that any perturbation will lead the economy off from the equilibrium path.* □

Proposition 2 says that we can not rule out the possibility of existence of indeterminacy if there exists $\delta \geq \mu$, such that $\max_{\theta\in\Theta} \left\|\mathbf{m}^{\sup^*}(\theta) - \mathbf{m}^{\inf^*}(\theta)\right\| > \eta$. Similarly, the application of proposition 3 is that there is no indeterminacy in the model if $\max_{\theta\in\Theta} \left\|\mathbf{m}^{\sup^*}(\theta) - \mathbf{m}^{\inf^*}(\theta)\right\| \leq \mu$, for any $\mu > 0$.

3.4 Numerical specifications

I apply the above algorithm to two examples, and illustrate the application of the sufficient condition for determinacy.

3.4.1 Example 1

I consider the parametrization studied by Kehoe and Levine (1990). More specifically, the preference is given by $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$. I choose $\beta = 0.5$, $\gamma = 4$, and $\{\mathbf{e}^a(s_{t+a})\}_{a=0}^2 = \{e^0, e^1, e^2\} = \{3, 12, 1\}$. There is only one short-lived bond θ available and the price is given by q_t.

As Kehoe and Levine (1990) pointed out this economy has three real steady states with prices and bond-holdings given by $q_1^* = 0.176$, $\theta_1^* = 5.772$, $q_2^* = 0.793$, $\theta_2^* = 3.732$, and $q_3^* = 44.634$, $\theta_3^* = 0.183$. They show that close to the middle steady state there must be a continuum of equilibria.

In our example, I start from a big set $\tilde{\mathbf{V}}_0(\theta) = \{c_{\min}^1 \leq m \leq c_{\max}^1\}$. I apply $\mathbf{B}^{h,\mu}$ to $\tilde{\mathbf{V}}_0(\theta)$ and end up with the area in the left panel of Fig. 1. The right panel represents the corresponding mapping from today's bond holding of middle age θ_t to tomorrow's holding θ_{t+1}.

As in this example, the maximum distance between the boundaries of the limit of the sequence $\tilde{\mathbf{V}}_{n+1}$ is given by a constant $d \gg \mu$. Proposition 2 suggests that we can not rule out the possibility of indeterminacy in this model economy.

3.4.2 Example 2

When the distribution of endowments is given by $\{\mathbf{e}^a(s_{t+a})\}_{a=0}^2 = \{e^0, e^1, e^2\} = \{3.5, 6, 1.5\}$, one can verify that there is only one real steady state. I apply the above algorithm for different values of μ. The algorithm always converges to the case that $\max_{\theta\in\Theta} \left\|\mathbf{m}^{\sup^*}(\theta) - \mathbf{m}^{\inf^*}(\theta)\right\| \leq \mu$. Proposition 3 implies that there may not exist indeterminacy in the model economy (Fig. 2).

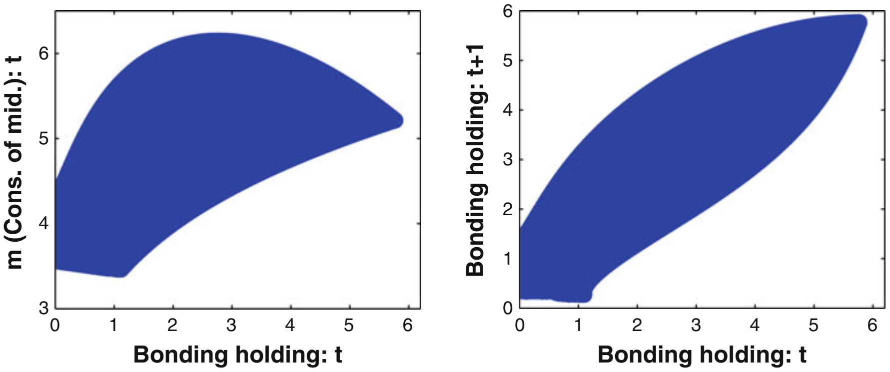

Fig. 1 The equilibrium set of economy 1

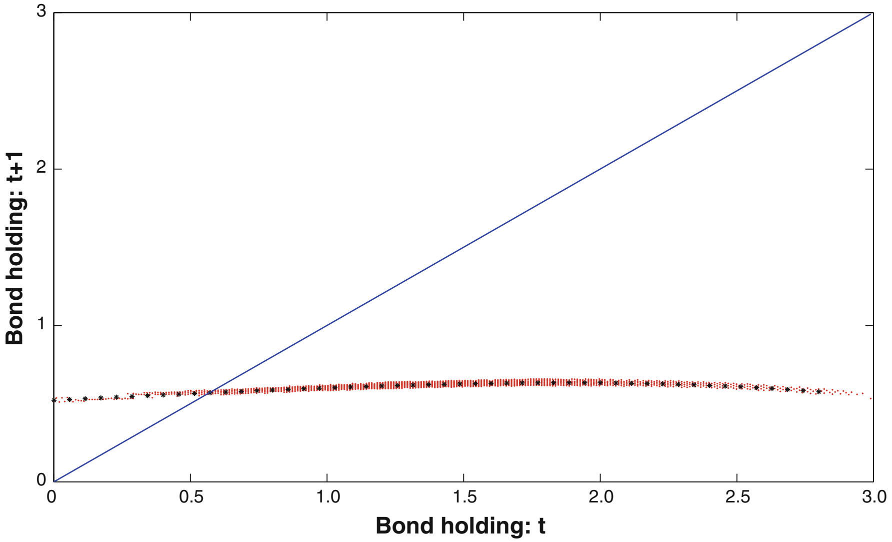

Fig. 2 The equilibrium set of economy 2

4 Simulation of OLG models with equilibrium indeterminacy

The correspondence $\tilde{\mathbf{V}}^*$ obtained in the previous section provides important information from which we can diagnose the existence of indeterminacy. However, this correspondence has limited application for us to conduct simulations. This is because it may contain non-equilibrium points. It would be misguiding to start the simulation from those non-equilibrium points. Therefore it is necessary to compute the exact equilibrium set, rather than the convex hull that contains the set. In this section, I fully discretize $\tilde{\mathbf{V}}^*$ and obtain an outer-approximation of $\mathbf{V}^*$ by constructing an operator $\mathbf{B}^{h,\mu,N}$, which is a discrete version of $\mathbf{B}^{\mathbf{h},\mu}$. It has been shown that the fixed point of the operator $\mathbf{B}^{h,\mu,N}$ converges to $\mathbf{V}^*$ uniformly (c.f. Feng et al. 2011). Please refer to the appendix for the details of the operator $\mathbf{B}^{h,\mu,N}$ (Fig. 3) .

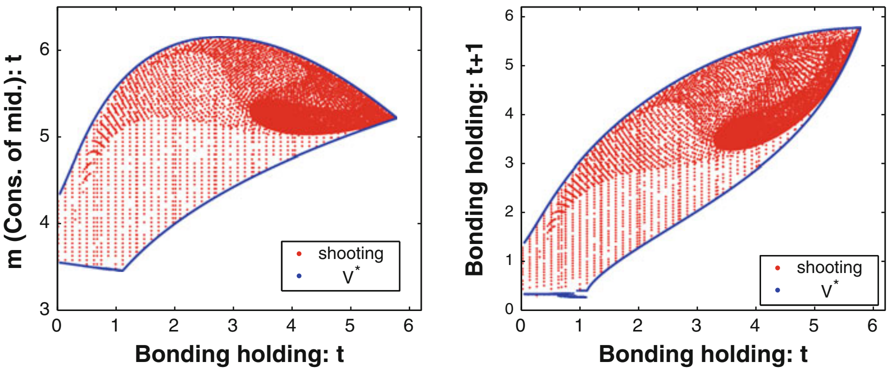

Fig. 3 The equilibrium set of economy 1 and the set computed by shooting method

Previous studies conjectured that a continuum of equilibria will all converge to the same steady state asymptotically (see Spear et al. 1990; Wang 1993). However this hypothesis has never been tested because there is no robust algorithm to compute the set of all equilibria for this type of economies. One contribution of this paper is that I test the validity of this statement in one example by computing the equilibrium set and conducting simulations. In what follows, I detail the simulation algorithm based on the equilibrium correspondence $\hat{\mathbf{V}}^{\mu,\varepsilon^*}$ obtained through the operator $\mathbf{B}^{h,\mu,N}$.

Assume that the economy begins with an initial bond-holding by the middle aged $\theta_{-1} \in \hat{\Theta}$ at time 0.

- At period $t = 0$, I pick an arbitrary $m_0 \in \hat{\mathbf{V}}^{\mu,\varepsilon^*}(\theta_{-1})$. Given $\{\theta_{-1}, m_0\}$, $\{q_0, \theta_0\}$ can be solved from the following equations

$$m_0 = e^1 + \theta_{-1} - q_0\theta_0 \tag{20}$$
$$q_0 \cdot u_c(m_0) = \beta u_c(e^2 + \theta_0), \tag{21}$$

  and then the value of m_1 is derived by using the first order condition for bond holding of the current young agent:

$$q_0 \cdot u_c(e^0 - q_0\theta_0) = \beta u_c(m_1). \tag{22}$$

- At period $t > 0$, I can solve for $\{q_t, \theta_t\}$ at given $\{\theta_{t-1}, m_t\}$ from equations

$$m_t = e^1 + \theta_{t-1} - q_t\theta_t \tag{23}$$
$$q_t \cdot u_c(m_t) = \beta u_c(e^2 + \theta_t), \tag{24}$$

  and infer m_{t+1} from

$$q_t \cdot u_c(e^0 - q_t\theta_t) = \beta u_c(m_{t+1}). \tag{25}$$

- Repeat this procedure to generate the sequence of $\{\theta_t, q_t\}_{t=0}^{T}$.

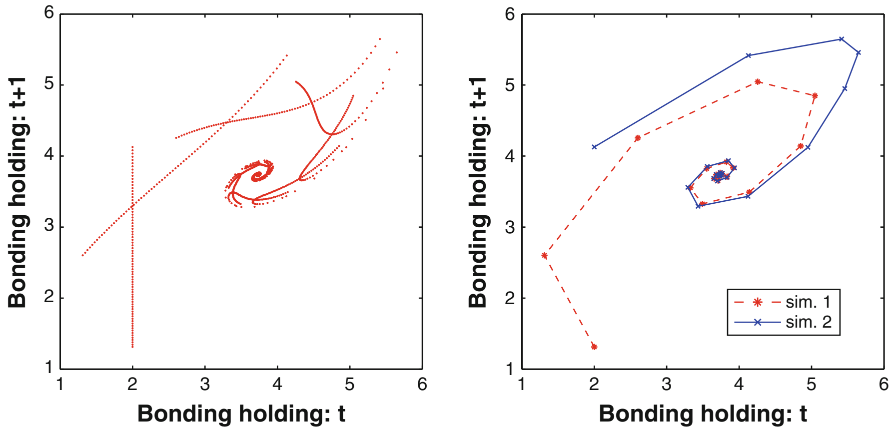

Fig. 4 Simulations of economy 1 with different initial conditions

For the economy described in Example 1, I start with $\theta_{-1} = 2.0$. As we can see from the left panel of Fig. 4, all simulated paths lead to the steady state $\theta^* = 3.73238$ as long as we pick $m_0 \in [4.0182, 6.0364] \subseteq \hat{\mathbf{V}}^{\mu,\varepsilon^*}(\theta_{-1} = 2)$, which implies that $\theta_0 \in [1.3126, 4.1277]$.

The indeterminacy in this simulation exercise illustrates that existence of numerous equilibrium paths in a deterministic OLG model can be indexed by the initial conditions for the shadow value of investment in bond $m_0 \in \hat{\mathbf{V}}^{\mu,\varepsilon^*}(\theta_{-1})$, which gives rise to the prices. After that, the Euler equation will uniquely pin down the equilibrium path. More specifically, the temporary equilibrium conditions yield a unique value of m_{t+1} for any given θ_t, m_t as in equation (25), hence the sequence of $\{\theta_t, q_t, m_t\}$.

Now imagine there are two different economies with exactly the same initial condition θ_{-1}. If their parameterizations are given by the one described in example 2, then they will converge to the steady state with the same speed and the same volatility as there is only one path leading to the long-run equilibrium. However, they may behave quite different if there is indeterminacy as in the above economy. The freedom of choosing m_0 makes them distinct equilibrium paths that lead to the same steady state. As a matter of fact, the economy that chooses $m_0 = 4.0182$ (A henceforth) will converge to the steady state in 105 periods with $mean(\theta_t) = 3.7088$, and $mean(q_t) = 0.8328$. While the economy with $m_0 = 6.0364$ (B henceforth) will converge to the steady state in 109 periods with $mean(\theta_t) = 3.7799$, and $mean(q_t) = 0.7740$.[3]

I also provide the welfare analysis by computing the consumption equivalent variation (CEV) along the transition path. I quantify the welfare effect by asking how much consumption has to be increased for all generations in each date in order to equate expected utilities to a benchmark economy. Without loss of generality, I choose econ-

[3] As the referee pointed out, all variables converge to a constant, for a sufficiently long simulated series. The standard errors of the simulations should converge to zero. However we report different statistics in Table 1. This is because convergence can only be defined as $\|\theta_t - \theta^*\| \leq \varepsilon$, where ε is a small positive number subject to machine precision. We also have to stop the simulation in finite time. Furthermore, we want to highlight the differences transition paths with distinct initial conditions before the convergence has reached. Hence we choose $\varepsilon = 10^{-12}$.

Table 1 Simulation for OLG without uncertainty

	$SD(\theta)$	$Mean(\theta)$	$SD(p)$	$Mean(p)$	$Mean(u)$	$Max(ee)$
Simulation A	0.3644	3.7088	0.3959	0.8328	−0.0073	$1.52 \times 1e^{-12}$
Simulation B	0.3725	3.7799	0.1123	0.7740	−0.0071	$1.44 \times 1e^{-12}$

omy B as benchmark, in which the social planner chooses a lower price (higher interest rate) to achieve smoother consumption across generations along the transition path. There will be a welfare gain of 1.2 % when the economy switches from A to B. In other words, more volatile prices along the transition path will translate into a welfare loss equivalent to reducing consumption of all agents by 1.2 % before the economy enters the steady state.

5 Concluding remarks

In this paper, I study the indeterminacy of equilibria in OLG models. I adopt the numerical method developed in Feng et al. (2011) and provide a general approach to identify the existence of indeterminacy by computing the boundary of the equilibrium set of the above economy. I implement this method for an OLG economy with two different parameterizations. In one case, the upper boundary of the computed equilibrium set is always bigger than the lower boundary in the limit of the numerical procedure, which implies that we can not rule out the existence of indeterminacy. The indeterminacy was ruled out in another example as the upper boundary is identical to the lower one.

In order to derive the implication of the indeterminacy on the long-run behavior of the economy, I solve the model numerically by finding all Markov equilibria and propose a way to simulate the OLG economy with a continuum of equilibria. Numerical results suggest that the economy endowed with the same initial condition may converge to the long-run equilibrium along different paths, and with different volatility when the economy has equilibrium indeterminacy. It would be interesting to study whether uncertainty will bring extra indeterminacy into the model as the household may have the freedom to select the transition and policy functions from the equilibrium correspondence. I leave this for future research.

Acknowledgements I am very grateful to Felix Kubler for his advice and encouragement. I am indebted to Kenneth Judd, David Levine, Adrian Peralta-Alva, Walt Pohl, Manuel Santos, Karl Schmedders, Fabrizio Zilibotti for very helpful comments and insightful discussions. I also thank the editor and an anonymous referee for their valuable comments. I greatly acknowledge financial support from NCCR-FINRISK. Special thanks go to Johannes Brumm for his editorial help. All remaining errors are exclusively mine.

Appendix

Outer approximation of $\mathbf{V}^*$ with operator $\mathbf{B}^{h,\mu}$

1. At each given Θ^i, I set $\mathbf{m}_1^{\inf}(\theta) \equiv \mathbf{B}^{\mathbf{h},\mu}\left(\mathbf{m}_0^{\inf}(\theta)\right) = \mathbf{m}_0^{\inf}(\theta)$ if either

$$\min_{\substack{m\in\left[\mathbf{m}_0^{\inf}(\theta),\mathbf{m}_0^{\inf}(\theta)+\mu\right]\\ \theta\in\Theta^i,\ \ m_+(\theta_+)\in\tilde{\mathbf{V}}_0(\theta_+(m))}} \left\| q\cdot u_c(e^0-q\theta_+(m))-\beta\cdot u_c(m_+(\theta_+))\right\| \leq \epsilon \quad (26)$$

or $\mathbf{m}_0^{\inf}(\theta)+\mu>\mathbf{m}_0^{\sup}(\theta)$. If any of these two conditions does not hold, then I set $\mathbf{m}_1^{\inf}(\theta)\equiv \mathbf{B}^{\mathbf{h},\mu}\left(\mathbf{m}_0^{\inf}(\theta)\right)=\mathbf{m}_{\mathbf{0}}^{\inf}(\theta)+\mu$. A symmetric procedure can be used to define $\mathbf{m}_1^{\sup}(\theta)\equiv \mathbf{B}^{\mathbf{h},\mu}\left(\mathbf{m}_0^{\sup}(\theta)\right)$.
Notice, given θ and m, I can determine the values for q,θ_+ by solving the following functions.

$$m-(e^1-\theta+q\theta_+)=0 \quad (27)$$

$$q\cdot u_c(m)-\beta\cdot u_c(e^2-\theta_+)=0 \quad (28)$$

In case of no solution exists, the operator $\mathbf{B}^{\mathbf{h},\mu}$ skips the above procedure and set $\mathbf{B}^{\mathbf{h},\mu}\left(\mathbf{m}_0^{\inf}(\theta)\right)=\mathbf{m}_{\mathbf{0}}^{\inf}(\theta)+\mu$ when $m\in\left[\mathbf{m}_0^{\inf}(\theta),\mathbf{m}_0^{\inf}(\theta)+\mu\right]$, $\mathbf{B}^{\mathbf{h},\mu}\left(\mathbf{m}_0^{\sup}(\theta)\right)=\mathbf{m}_{\mathbf{0}}^{\sup}(\theta)-\mu$ when $m\in\left[\mathbf{m}_0^{\sup}(\theta)-\mu,\mathbf{m}_0^{\sup}(\theta)\right]$.

2. Repeat step 1 until the sequence of $\mathbf{m}_n^{\inf}(\theta)$, $\mathbf{m}_n^{\sup}(\theta)$ have converged to their limit $\mathbf{m}^{\inf^*}(\theta)$, $\mathbf{m}^{\sup^*}(\theta)$.

Note that $\mathbf{B}^{h,\mu}$ is monotone decreasing by construction, and generates a convergent sequence of convex-valued correspondences containing the equilibrium correspondence $\mathbf{V}^*$. At the limit of the procedure, I obtain the smallest possible convex hull $\tilde{\mathbf{V}}^*_{\theta\in\Theta^i}(\theta)$ that contains $\mathbf{V}^*_{\theta\in\Theta^i}(\theta)$, where $\tilde{\mathbf{V}}^*_{\theta\in\Theta^i}(\theta) := \Big\{ m(\theta)|\theta \in \Theta^i$, $m\in\left[\mathbf{m}^{\inf^*}(\theta),\mathbf{m}^{\sup^*}(\theta)\right]\Big\}$. Finally I have $\tilde{\mathbf{V}}^*=\cup_i\tilde{\mathbf{V}}^*_{\theta\in\Theta^i}(\theta)$.

Discretization of operator $\mathbf{B}^{h,\mu}$

I then use $\tilde{\mathbf{V}}^*$ as initial condition for an operator on discrete correspondences defined as follows:

The vector of possible values for bond-holding and shocks are given by $\hat{\Theta}=\left\{\theta_0^i\right\}_{i=1}^{N_\theta}$, and for each θ_0^i, I also define a finite vector of possible values for $\hat{\mathbf{V}}_0^{\mu,\varepsilon}\left(\theta_0^i\right)=\left\{m_0^{i,j}\right\}_{j=1}^{Nv}$. Notice, $\lim_{N_\theta\to\infty}\hat{\Theta}=\Theta$, $\lim_{Nv\to\infty}\hat{\mathbf{V}}_0^{\mu,\varepsilon}\left(\theta_0^i\right)=\tilde{\mathbf{V}}_0^{\mu,\varepsilon}\left(\theta_0^i\right)$. Finally, I construct the discrete version of operator $\mathbf{B}^{h,\mu,N}$ by eliminating points that cannot be continued (in the Euler equation, for a pre-determined tolerance $\epsilon>0$) as follows:

1. Given θ_0^i, pick a point $m_0^{i,j}$ in the vector $\hat{\mathbf{V}}_0^{\mu,\varepsilon}\left(\theta_0^i\right)$. From $m_0^{i,j}$ I can determine the values of $\left(\theta_+^{i,j},q^{i,j}\right)$ by solving for

$$m_0^{i,j}-\left(e^1+\theta_0^i-q^{i,j}\theta_+^{i,j}\right)=0. \quad (29)$$

$$q^{i,j}\cdot u_c\left(m_0^{i,j}\right)-\beta u_c\left(e^2+\theta_+^{i,j}\right)=0 \quad (30)$$

Thus, if for all $m_+ \in \hat{\mathbf{V}}_0^{\mu,\varepsilon}(\theta_+^{i,j}, s_+) = \left\{m_+^l(\theta_+^{i,j}, s_+)\right\}_{l=1}^{N_V}$ I have

$$\min_{m_+ \in \{m_+^l\}_{l=1}^{N_V}} \left\| q^{i,j} \cdot u_c \left(e^0 - q^{i,j}\theta_+^{i,j}\right) - \beta u\left(m_+\right) \right\| > \epsilon \tag{31}$$

then $\hat{\mathbf{V}}_1^{\mu,\varepsilon}\left(\theta_0^i\right) = \hat{\mathbf{V}}_0^{\mu,\varepsilon}\left(\theta_0^i\right) - m_0^{i,j}$.

2. Iterate over all possible values $m_0^{i,j} \in \hat{\mathbf{V}}_0^{\mu,\varepsilon}\left(\theta_0^i\right)$, and all possible $\theta_0^i \in \hat{\Theta}$.
3. Iterate until convergence is achieved $\sup \left\| \hat{\mathbf{V}}_n^{\mu,\varepsilon} - \hat{\mathbf{V}}_{n-1}^{\mu,\varepsilon} \right\| = 0$.

At the limit of the above algorithm, I have $\lim_{n\to\infty} \hat{\mathbf{V}}_n^{\mu,\varepsilon} = \hat{\mathbf{V}}^{\mu,\varepsilon^*}$.

I apply the operator $\mathbf{B}^{h,\mu,N}$ to the model economy in Example 1. I use $\tilde{\mathbf{V}}_{n+1}$ as the initial condition for our iteration procedure. Our numerical result indicates that $\tilde{\mathbf{V}}_{n+1}$ is the fixed point of $\mathbf{B}^{h,\mu,N}$. As shown in the left panel of Fig. 3, the area between two solid lines represent the approximate solution. I also include the solution obtained from backward shooting algorithm in the same graph, which is presented by dots.

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