Numerical Simulation of Nonoptimal Dynamic Equilibrium Models*

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Abstract

In this paper we propose a recursive equilibrium algorithm for the numerical simulation of nonoptimal dynamic economies. This algorithm builds upon a convergent operator over an expanded set of state variables. The fixed point of this operator defines the set of all Markovian equilibria. We study approximation properties of the operator. We also apply our recursive equilibrium algorithm to various models with heterogeneous agents, incomplete financial markets, endogenous and exogenous borrowing constraints, taxes, and money.

Keywords: Heterogeneous agents, real and financial frictions, sequential competitive equilibrium, Markov equilibrium, computation, simulation, accuracy.

JEL Codes: C6, D5,E2.

*An earlier version of this paper was circulated as "Existence and Computation of Markov Equilibria for Dynamic Nonoptimal Economies" by J. Miao and M. Santos, March 2005. We thank Chanont Banternghansa and Jan Auerbach for their computational assistance. We are grateful to two anonymous referees and an editor for very helpful comments.

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1 Introduction

In this paper we propose a reliable *recursive equilibrium algorithm* for the numerical simulation of nonoptimal dynamic economies, and study its convergence and accuracy properties. Numerical simulation of these economies is usually a formidable task because of various technical issues that preclude direct application of standard dynamic programming techniques. We apply our numerical algorithm to various models with heterogeneous agents and real and financial frictions. The quantitative analysis of these models becomes critical to advance our understanding in several basic areas of macroeconomics and finance.

Standard solution methods search for a continuous equilibrium function over a natural state space of exogenous and endogenous state variables. Since the seminal work of Kydland and Prescott (1980), it is well known that equilibria of nonoptimal economies may not admit a recursive representation over this natural state space. These authors consider a game of optimal taxation, and rewrite their model in a recursive form by appending a Lagrange multiplier to the original state space so as to characterize the exact solution. Their simple model comprises a representative household, and the set of continuation Lagrange multipliers is unique. This uniqueness property is a rather limiting condition for many other economies.

Our recursive equilibrium algorithm applies to a broad range of dynamic competitive-markets economies. We consider an abstract framework, and provide a characterization of Markovian equilibrium representations towards the numerical simulation of these economies. While some characterizations of Markovian equilibria for nonoptimal economies are available, these characterizations are model-dependent. Moreover, the numerical implementation of the proposed algorithms together with their approximation properties have never been analyzed in the literature.

Numerical simulation of nonoptimal economies by standard solution methods may result in substantial approximation errors. For instance, we simulate below a simple overlapping generations (OLG) model by an established algorithm under a continuous equilibrium func-

tion. The computed solution may present large approximation errors, and fail to mimic the true dynamics. In spite of these large approximation errors, the algorithm can be quite deceptive as it produces small Euler equation residuals, or may do well under some other independent accuracy checks. Peralta-Alva and Santos (2010) discuss some of the pitfalls in the computation of equilibrium solutions for an economy with distortionary taxation.

Positive results on existence of a continuous equilibrium over a natural state space rely upon certain monotonicity properties of the equilibrium dynamics [e.g., see Bizer and Judd (1989), Coleman (1991), and Datta, Mirman and Reffett (2002)]. For the canonical one-sector growth model with taxes and externalities, monotone dynamics follow from fairly mild restrictions on the primitives. But monotone dynamics are much harder to obtain in multi-sector models with heterogeneous agents and real and financial frictions.

Several papers are concerned with the characterization of recursive equilibria for nonoptimal economies. As already pointed out, these characterizations are model-dependent, and do not consider numerical implementation and approximation properties of these algorithms. Abreu, Pierce, and Stacchetti (1990) introduce continuation utility values to find a recursive representation of sequential equilibria for dynamic games. This characterization of equilibrium seems quite natural in repeated games, but it may become computationally demanding in some other models. Duffie *et al.* (1994) search for general representations of stationary equilibria over an expanded state space that includes all endogenous variables such as asset prices and individual consumptions. Again, expanding the state space over all state variables may slow down the computation process. Building upon these methods, Kubler and Schmedders (2003) show existence of a Markovian equilibrium for a class of financial economies with collateral requirements. Their computations are based on a projection-type algorithm iterating in the space of continuous functions. This computational procedure cannot guarantee convergence to the true solution. Marcet and Marimon (2010) study a general class of contracting problems with incentive constraints. Following Kydland and Prescott (1980), they enlarge the state space with a vector of weights for the

utility of each agent, and compute a transition for such weights from the shadow values of the agents' participation constraints. They assume that equilibrium solutions can be characterized by convex social planning problems. By construction this method cannot capture multiple equilibria, but seems to be more operative for the computation of various dynamic incentive problems written in a Pareto-welfare form.

Our work is closest to Kubler and Schmedders (2003), but we consider a broader family of economies that may include endogenous and exogenous borrowing constraints. In the numerical implementation, we discretize our algorithm to preserve its convergence properties. Thus, unlike Kubler and Schmedders (2003), we iterate over candidate equilibrium sets – rather than functions – to guarantee convergence to the original equilibrium set. We can thus compute the set of all competitive equilibria. As discussed below, this reliable discretization procedure can successfully be applied to various types of models, and it seems particularly useful for OLG models and some other infinite-horizon models with various types of real and financial frictions.

Section 2 considers two simple examples intended to highlight some major computational issues and the workings of our algorithm. Section 3 introduces our framework of analysis. We provide a general characterization of Markov equilibria for nonoptimal economies. The set of Markov equilibria is computed as the fixed-point of a monotone operator embedding all *short-run* equilibrium conditions. This operator has good convergence and stability properties, and hence it provides the foundations for the formulation of our reliable recursive equilibrium algorithm. Section 4 studies the numerical implementation of our algorithm and its approximation properties. We apply these numerical procedures to two types of models. Sections 5 is devoted to the numerical simulation of a simple OLG model, and Section 6 considers a model of international trading with various market frictions. We conclude in Section 7.

2 Two Illustrative Examples

2.1 An Overlapping Generations Economy

Time is discrete, $t = 0, 1, 2, \cdots$. At each date t a new consumer appears in the economy. Each consumer receives an endowment e_1 of a perishable good when young, and e_2 when old. There is a single asset called money that can be held for trading. This asset pays zero dividends, and it belongs to the initial old generation starting the economy. The money supply M remains constant over time.

Let P_t be the price of money in terms of the aggregate good at date t. Then, a typical consumer born in period t solves the following optimization problem:

$$\max u(c_t^t) + v(c_{t+1}^t)$$

$$s.t.$$

$$c_t^t + \tfrac{M_t}{P_t} = e_1$$

$$c_{t+1}^t = e_2 + \tfrac{M_t}{P_{t+1}}.$$

Note that c_s^t denotes consumption at date s of the agent born at time t, for $s = t, t+1$, and M_t is the amount of money demanded at time t.

A sequential competitive equilibrium for this economy is a sequence of prices $\{P_t\}$ and sequences of optimal choices $\{c_t^t, c_{t+1}^t, M_t\}$ for the given prices, such that both commodity and money markets clear at all times:

$$c_t^t + c_t^{t-1} = e_1 + e_2$$

$$M_t = M,$$

for all $t \geq 0$. For interior solutions, under the concavity of the objective function, the budget-constrained optimal choice $\{c_t^t, c_{t+1}^t, M_t\}$ is fully characterized by the first-order

conditions:

$$u'(c_t^t) = \lambda_t$$
$$v'(c_{t+1}^t) = \lambda_{t+1}$$
$$\tfrac{\lambda_t}{P_t} = \tfrac{\lambda_{t+1}}{P_{t+1}},$$

where λ_t is a Lagrange multiplier at time t.

To analyze the dynamics of the model, we can indistinctly consider any of the following three (state) variables: Consumption, c_t^t, the price level, P_t, or the amount of real money balances, $b_t \equiv M/P_t$. That is, all these three variables provide the same information. Then, rearranging all the above equations, equilibrium sequences $\{c_t^t, c_{t+1}^t, M_t, P_t\}$ can be fully characterized by the equation:

$$b_t u'(e_1 - b_t) = b_{t+1} v'(e_2 + b_{t+1}). \tag{1}$$

A standard approach for computing equilibrium solutions would be to search for a continuous function $g : X \to X$ with $b_{t+1} = g(b_t)$ for all $t \geq 0$ and

$$b_t u'(e_1 - b_t) = g(b_t) v'(e_2 + g(b_t)). \tag{2}$$

We would like to stress that existence of a continuous equilibrium function $b_{t+1} = g(b_t)$ requires further assumptions on the model primitives. More specifically, a continuous equilibrium function $b_{t+1} = g(b_t)$ occurs under monotone equilibrium dynamics: An upward sloping offer curve arising under the assumption of gross substitutes in consumption. But if the offer curve is backward bending, then $b_{t+1} = g(b_t)$ is just a correspondence, which may not have a continuous selection.

For instance, as is well known [Grandmont (1985)] the offer curve is backward bending for the following parameterization:

$$u(c) = c^{0.45},\ v(c) = -\frac{0.8}{7}c^{-7},\quad M_0 = 1, e_1 = 2, e_2 = 2^{6/7} - 2^{1/7}.$$

See Figure 1. Here, the upper and lower arms are two continuous equilibrium selections. As illustrated in Section 5, there are other cases in which no continuous equilibrium selection does exist.

For this parameterization, we applied a version of the projection method over (2) to compute a continuous policy $b_{t+1} = g(b_t)$. Depending on the initial guess, the numerical approximation converges either to the upper or the lower arm of the offer curve, or to some other hybrid solution. This strong dependence on initial conditions is a rather undesirable feature of this computational method. In particular, if we only consider the lower arm of the actual equilibrium correspondence then all competitive equilibria converge to autarchy (zero monetary holdings). But if we iterate over the upper arm of the offer curve, we find that money holdings converge monotonically to the stationary solution $\frac{\bar{M}}{p} = 0.4181$. Hence, even in the deterministic version, we need a global approximation of the equilibrium correspondence to analyze the various predictions of the model. As a matter of fact, none of these two selections would capture a two-period equilibrium cycle in which real money holdings oscillate between 0.8529 and 0.0953 (see Figure 2). It is also known that the model has a three-period cycle.

As shown in Section 5, for certain parameterizations an OLG economy may not admit an equilibrium function over the natural space of state variables. To compute the equilibrium set, we could consider additional auxiliary variables. One possible choice is to select continuation utilities over the multiple equilibrium paths. Continuation utilities, however, will force us to discard the first-order condition (1). Thus, from a computational point of view it seems optimal to build an efficient numerical algorithm based upon (1).

Let us then define m_{t+1} as $m_{t+1} = \frac{b_{t+1}}{b_t} v'(e_2 + b_{t+1})$. Now, equation (1) reduces to $u'(e_1 - b_t) = m_{t+1}$. This simple equation seems much easier to compute. Accordingly, we propose to compute the set of all equilibrium paths over an expanded state space (b_t, m_t). In this expanded state space we will define an equilibrium correspondence that generates all equilibrium paths. With this background in place, let us further illustrate our computational

method in the following example.

2.2 Optimal Growth

Consider the infinite-horizon optimization problem

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$s.t.$$

$$c_t + k_{t+1} = F(k_t, 1) + (1-\delta)k_t$$

$$\text{given } k_0, 0 < \beta < 1, 0 < \delta \leq 1.$$

Under standard conditions for u and F, the solution to the above problem can be fully characterized by the (infinite) set of Euler equations:

$$u'(F(k_t,1) + (1-\delta)k_t - k_{t+1}) = \beta u'(F(k_{t+1},1) + (1-\delta)k_{t+1} - k_{t+2})(F_k(k_{t+1},1) + 1 - \delta).$$

A common approach is to search for a continuous function $k_{t+1} = g(k_t)$ over this time-homgeneous non-linear system:

$$u'(F(k_t,1) + (1-\delta)k_t - g(k_t)) = \tag{3}$$

$$\beta u'(F(g(k_t),1) + (1-\delta)g(k_t) - g(g(k_t)))(F_k(g(k_t),1) + 1 - \delta)$$

$$\text{for all } t \geq 0.$$

Under some specifications for the production function F, a continuous solution g may not exist [cf. Boldrin and Rustichini (1994)]. For instance, in models with externalities, function F may be written as $F(k,1) = f(\hat{k}, k, 1)$, with $\hat{k} = k$ at every equilibrium solution. As a matter of fact, in non-convex programming the Euler equation may pick sub-optimal solutions. In those cases, the set of optimal solutions may be characterized by continuation utilities or some other auxiliary variables.

For simplicity, let us assume that the system of Euler equations (3) determines all equilibrium solutions. Then, we cannot hope to find a recursive representation of equilibrium

by conditioning on variable k only. Indeed, for every k_0 there is a continuum of vectors (k_1, k_2) that satisfy the above Euler equation (3). A recursive characterization of equilibrium, however, can readily be obtained by letting the state space comprise equilibrium pairs (c, k). That is, for each (k_0, c_0) the resource constraint determines k_1; further, c_1 can be solved from the Euler equation. Therefore, for each equilibrium pair (c, k) the Euler equation (3) generates a unique continuation value (k_+, c_+). We would like to stress that for computational purposes it may be more operative to expand the state space with auxiliary variable $m \equiv u'(c)(F_k(k, 1) + 1 - \delta)$, i.e. the shadow return of one unit of investment. As in the preceding example, the Euler equation is linear in m. This will be useful in the computation stage.

Let K be the domain of possible values for the capital stock and M the set of possible values for m. We could start with space $K \times M$ as an initial guess for all starting equilibrium values. Usually, this universal set is too broad: Many pairs (k, m) may lack continuation values (k_+, c_+) over the above Euler equation (3). Each initial guess will be refined under the action of the following operator B embedding all *short-run* equilibrium conditions.

Let $V : K \to M$ be a large enough correspondence of potential continuation values (k_+, m_+). For every k, let $m \in BV(k)$ if there exists (c, k_+, m_+) with $m_+ \in V(k_+)$ such that

$$c + k_+ = F(k, 1) + (1 - \delta)k$$
$$u'(c) = \beta m_+.$$

where $m \equiv u'(c)(F_k(k, 1) + 1 - \delta)$. Correspondence V is chosen large enough[1] so that the new correspondence $B(V)$ is a subset of V. Then, by construction we obtain a decreasing sequence of correspondences $V_{n+1} = B(V_n)$ that converge to the equilibrium correspondence V^*. Therefore, starting from each pair $(k, m) \in graph(V^*)$ we can generate a sequence of equilibrium solutions $\{c_t, k_{t+1}\}$ satisfying the above equation system at all times. As a

[1] Our method works under the weaker condition that V contains equilibrium correspondence V^* presently defined.

matter of fact, every sequential equilibrium solution can be generated under some initial equilibrium pair $(k, m) \in graph(V^*)$.

To summarize, under the action of operator B, the *recursive equilibrium algorithm* finds a Markov equilibrium correspondence V^* that can generate all (sequential) equilibrium solutions.

There are three main points to be emphasized from this exercise. First, the equilibrium correspondence is the maximal fixed point of operator B. That is, $V^* = B(V^*)$ and $V' \subset V^*$ for any other fixed point $V' = B(V')$. Hence, under a proper formulation of the state space the existence of a fixed point V^* is tantamount to the existence of a sequential equilibrium solution. Second, the iteration process under operator B proceeds over correspondences rather than functions. While iteration over correspondences may be computationally more costly, the recursive equilibrium algorithm guarantees convergence to the set of equilibrium solutions under a good initial guess V_0. And third, it is important to keep in mind that the recursive equilibrium algorithm is subject to the curse of dimensionality, as it may involve maximizations and integrations over spaces of functions, and set iterations. Indeed, some characterizations of Markov equilibria may not be computable. Therefore, the choice of the state space is usually critical. In the previous two examples, the state space has been enlarged with the shadow values of investment.

3 General Theory

In this section, we first set out a general analytical framework that encompasses various recursive economic models. Their equilibrium time series, however, may depend on full histories of shocks and economic variables. Therefore, these equilibria are not directly amenable to computation unless we can find a Markovian representation over a well chosen state space. Then, we present a formal version of our recursive equilibrium algorithm. In Section 4 we develop a convergent numerical algorithm with desirable approximation properties.

Following Ljungqvist and Sargent (2000), a recursive equilibrium representation is conformed by "a *transition* mapping the state of the model today into the state tomorrow and a *function* mapping the state into the other endogenous variables of the model." Duffie *et al.* (1994) show that under fairly general conditions it is possible to provide a recursive representation of sequential competitive equilibria by expanding the state space with all endogenous variables. Their approach does not cover models with endogenous constraints – nor does it provide a way to find or approximate equilibria. Our analysis will be guided by computational considerations, and so it is imperative to keep a manageable state space.

3.1 General Framework

Time is discrete $t = 0, 1, 2, \cdots$. At every date t the economy is composed of I agents, and it is shocked by a vector of exogenous variables z. This vector follows a Markov chain $(z_t)_{t\geq 0}$ over a finite set $Z = \left\{1, 2, ..., \widehat{Z}\right\}$ as described by transition probabilities $\pi(z'|z)$ for all $z, z' \in Z$. The initial state, $z_0 \in Z$, is known to all agents in the economy. Then $z^t = (z_0, z_1, z_2, ..., z_t) \in Z^{t+1}$ is a history of shocks, often called a date-event or node. Endogenous predetermined variables are denoted by x, with $x \in X$, $X \subset \mathbb{R}^N$. Vector x may include agents' holdings of physical capital, human capital, and financial assets. All other endogenous variables are denoted by y, with $y \in Y$, $Y \subset \mathbb{R}^L$. Vector y may include equilibrium prices, choice variables such as consumption and investment, and auxiliary variables such as Lagrange multipliers, shadow values of investment and continuation utilities. Indeed, certain auxiliary variables may either be necessary or may allow for a more operational representation of equilibrium.

In Section 6 below, our set of auxiliary variables includes shadow values of investment of each existing asset for every agent, $m \in M$, $M \subset \mathbb{R}^K$, and continuation utilities, $p \in \mathbb{R}^I$, as is common in the literature on incentive constraints. Agents will have the choice to default. It is thus necessary to specify the payoff of default, which in our case implies permanent exclusion from commodities and financial markets. Default carries a lifetime utility that

may depend on the vectors of shocks z and endogenous predetermined state variables x. When non-convexities arise from individual effects on the aggregate state variables, first-order conditions cannot longer be invoked. Hence, computations must consider global maximization methods. More precisely, each of the I agents in this economy confronts an expected discounted lifetime utility given by a function $P^{aut} : \mathbb{R}^N \times Z \to \mathbb{R}^I$ in case of default. This payoff function P^{aut} may depend on both individual and aggregate state variables, and may give rise to a non-concave individual optimization problem.

The thrust of our analysis is the computation of sequential competitive equilibria (SCE), as described by infinite sequences $\{x(z^t), y(z^t)\}_{t\geq 0}$. We limit this exercise to models where all SCE lie in a compact space and can be characterized by aggregate resource constraints and *short-run* optimality conditions involving only variables of two contiguous time periods, t and $t+1$. Specifically, the law of motion of the state vector x is conformed by a system of non-linear equations:

$$\varphi\left(x_{t+1}, x_t, y_t, z_t\right) = 0. \tag{4}$$

Function φ may embed technological constraints as well as individual budget constraints. For some models we can explicitly solve for x_{t+1} as a function of (x_t, y_t, z_t). But in some other applications such as in models with adjustment costs, x_{t+1} may not admit an analytical solution.

Further, a SCE $\{x(z^t), y(z^t)\}_{t\geq 0}$ must satisfy an infinite sytem:

$$(x_t, y_t, z_t, x_{t+1}, y_{t+1}, z_{t+1}) \in \Phi \tag{5}$$

for all $t \geq 0$. Functional Φ describes various *short-run* equilibrium conditions: (i) Euler equations, in which case Φ is simply a non-linear system, (ii) one-period ahead constrained-optimization to account for non-concave maximization programs because of real and financial distortions and additional participation constraints, (iii) market-clearing conditions, and (iv) various types of budget restrictions and resource constraints.

We say that a model is *recursive* and *time invariant* if there exist functionals φ and

Φ characterizing the set of all SCE under conditions (4-5). Several assumptions underlie this abstract formulation.

First, the space of endogenous variables $X \times Y$ is compact. Hence, transversality conditions at infinity are usually trivially satisfied. Section 6 below shows how a compact domain $X \times Y$ may arise from optimization conditions in the presence of unbounded utility and production functions. Therefore, (4-5) must provide a set of sufficient conditions for the characterization of all SCE. Second, (4-5) only involve variables at times t and $t+1$. Hence, production and utility functions, technological, borrowing, and incentive constraints, must satisfy certain intertemporal separability assumptions. For instance, some forms of habit formation may be incorporated in the analysis by including auxiliary variables. Third, (4-5) are time invariant. That is, we search for a time-homogeneous Markovian representation of SCE, which will be given by an equilibrium correspondence mapping current states into equilibrium values for each successor node.

3.2 The Recursive Equilibrium Algorithm

The set of SCE may not admit a recursive representation over the standard state space comprising exogenous shocks z and predetermined endogenous variables x. To recover a recursive structure it is necessary to enlarge the state space. The required expansion of the state space will depend on the economic application. Hence, at this stage of our analysis we will simply assume that the equilibrium values of the required vector of auxiliary variables are described by an equilibrium correspondence $V^* : (x, z) \longmapsto V^*(x, z) \subseteq Y$. This equilibrium correspondence may contain discontinuities and multiple equilibria. Under standard continuity conditions on utility and production functions, the equilibrium correspondence is usually upper semicontinuous.

The theoretical underpinnings of our *recursive equilibrium algorithm* rest on the iteration of monotone equilibrium inclusions [Kydland and Prescott (1980) and Abreu, Pierce and Stacchetti (1990)] that lead to a convergent process. We first select an appropriate set

of state variables, and a well chosen initial correspondence V_0. Then, we apply a monotone operator, B, that generates sequences of non-empty decreasing compact sets $\{V_n\}$ shrinking to the equilibrium correspondence V^*. Operator B embodies all *short-run* equilibrium conditions (4-5) from any initial value z to all immediate successor nodes z_+. This operator is analogous to the expectations correspondence of Duffie *et al.* (1994), albeit it may contain a smaller set of endogenous variables. Using operator B, we can generate the set of all SCE using time-invariant equilibrium selections.

More precisely, for any given V under the action of operator B we obtain $V' = B(V)$. Correspondence V' is defined as follows. Pick a vector (x, z). Then, $v \in V'(x, z)$ if there is a vector $(y, x_+, y_+(z_+),\ v_+(z_+))$ for all $z_+ \in Z$, with $v_+(z_+) \in V(x_+, z_+)$ such that the resulting vector $(x, y, , z, x_+, y_+, z_+)$ satisfies the temporary equilibrium conditions (4-5).

For models where a SCE exists, correspondence $B(V)$ will be non-empty provided that our initial guess V_0 has been properly chosen. Note that by construction operator B is monotone: If $V \subset \hat{V}$ then $B(V) \subset B(\hat{V})$.[2] Further, under standard continuity conditions on functionals φ and Φ it follows that if V has a closed graph then $B(V)$ will have a closed graph.

Assumption 3.1 *Operator B preserves compactness: If V is compact valued then $B(V)$ is also compact valued.*

Assumption 3.1 will allow us to establish some uniform convergence properties of the algorithm. This assumption is not necessary to show existence of a fixed-point solution V^* and the global convergence of the iteration process.

Theorem 3.1 *(existence and global convergence) Let V_0 be a compact-valued correspondence such that $V_0 \supset V^*$. Let $V_n = B(V_{n-1})$ for all $n \geq 1$. Then, operator B has a fixed-point solution, i.e., $V^* = B(V^*)$, where $V^* = \lim_{n\to\infty} V_n$. Moreover, V^* is the largest fixed point of operator B, i.e., $V = B(V)$ implies $V \subset V^*$.*

[2]For correspondences $V, \hat{V}$ we say that $V \subset \hat{V}$ if $V(x, z) \subset \hat{V}(x, z)$ for all (x, z). We shall consider the usual notion of distance over sets given by the Hausdorff metric.

We again would like to remark that operator B iterates over sets rather than functions. Hence, if there are multiple equilibria we can find all of them. By definition, for any $(x, z, v) \in graph(V^*)$, under the action of operator B we can generate a new vector (x_+, z_+, y, v_+) that can be extended into a SCE $\{x(z^t), y(z^t)\}_{t\geq 0}$. Since the fixed point of operator B is an upper semicontinuous correspondence, it is possible to select a measurable policy function $y = g^y(x, z, v)$, a transition function $v_+(z_+) = g^v(x, z, v; z_+)$, and continuation values for the endogenous predetermined variables x_+ so that $\varphi(x_+, x, y, z) = 0$. Let us summarize these future equilibrium values over the extended state space as $g(x, z, v; z_+) = (x_+, z_+, v_+)$. Then, g is a Markovian equilibrium selection.[3]

To summarize, the set of SCE $\{x(z^t), y(z^t)\}_{t\geq 0}$ admits a recursive representation in an expanded state space. Our *recursive equilibrium algorithm* rests upon iteration of monotone sets under an operator B. For a well chosen initial correspondence, the iteration process converges to the Markov equilibrium correspondence V^*. We now proceed to the numerical implementation of the algorithm and to study its approximation properties.

4 Numerical Implementation

In this section we develop a numerical implementation of operator B and study its convergence and accuracy properties. For models with unique equilibria, our results imply that the accuracy of the numerical approximation is of the same order of magnitude as the mesh size of our discretization. For models with multiple equilibria, the fixed point of the numerical algorithm *converges uniformly* to the Markov equilibrium correspondence as the mesh size of the discretization converges to zero.

For dynamic games, Judd, Yeltekin and Conklin (2003), and Judd and Yeltekin (2010), develop an approximation procedure with good accuracy properties. Essentially, their approximation method works well for convex equilibrium correspondences. The convexity

[3]It should be clear that $g(\cdot; z_+)$ denotes a coordinate function of $g(\cdot)$ corresponding to the successor node $z_+|z$.

of the equilibrium correspondence may be achieved via a public randomization device. Randomization over the original set of strategies seems quite appealing in game theoretic settings. Such *ex post* convexification, however, may arbitrarily expand the equilibrium set of a competitive economy, and may not be compatible with individual optimization behavior. Note that by construction operator B is monotone, and maps compact sets into compact sets, but it does not preserve convexity.

We now proceed as follows. First, we partition the state space into a finite set of J simplices with mesh size h. Compatible with this partition we consider a sequence of *step correspondences*, which take constant set-values on each simplex. Step correspondences are the analog of step functions, and have good approximation properties. We also introduce a finite-dimensional outer approximation over the image of the step correspondences; this outer approximation is made up of Σ cubes or finite-dimensional elements. Then, combining these approximations we obtain a computable operator $B^{h,\Sigma}$ with accuracy parameters (h, Σ). Under the action of operator $B^{h,\Sigma}$, we construct a sequence of correspondences that converge to a fixed point containing equilibrium correspondence V^*. We shall study accuracy properties of the algorithm as we refine our discretizations over (h, Σ).

4.1 The Numerical Algorithm

Assume that all equilibrium state vectors (x, z, v) belong to some set S, which is a subset of the product space $S = X \times Z \times Y$. Let $\left\{X^j\right\}_{j=1}^{J}$ be a finite family of simplices with non-empty interior such that $\cup_j X^j = X$ and $int(X^j) \cap int(X^{j'})$ is empty for every pair $X^j, X^{j'}$. Define the *mesh size* h of this discretization as

$$h = \max_j diam\left\{X^j\right\}.$$

For any multivalued mapping $V : X \times Z \to 2^Y$, where 2^Y denotes the subsets of vectors for space Y containing the required auxiliary variables, an approximation V^h compatible with the partition $\left\{X^j\right\}$ takes on constant set-values $V^h(x, z)$ on each simplex X^j. More

precisely,

$$V^h(x,z) = \cup_{x \in X^j} V(x,z), \text{ for each given } z \text{ and all } x \in X^j. \tag{6}$$

This definition of step correspondence V^h will include all equilibrium values and preserve the monotonicity property over the discretized process. Analogously, over each X_j we define operator $B^h(V)$ as $B^h(V)(x,z) = \cup_{x \in X^j} B(V)(x,z)$, for each given z and all $x \in X^j$. As before, one can prove that B^h has a fixed-point solution. To obtain a computable representation of these correspondences we also discretize the image space. For a given set V we say that $\mathcal{C}^\Sigma(V) \supseteq V$ is an Σ-element outer approximation of V if $\mathcal{C}^\Sigma(V)$ can be generated by Σ elements. We require this numerical representation to preserve monotonicity: $V \subset \hat{V}$ implies $\mathcal{C}^\Sigma(V) \subset \mathcal{C}^\Sigma(\hat{V})$. This is essential to guarantee monotonicity of a computable version of operator B. We also require $\lim_{\Sigma \to \infty} \mathcal{C}^\Sigma(V) = V$.

Combining these approximations, we can construct a new operator $B^{h,\Sigma}$ as follows. We first define the step correspondence $B^h(V)$ of $B(V)$. Then, each set-element of $B^h(V)$ is adjusted by the Σ-element outer approximation to get $\mathcal{C}^\Sigma\left(B^h(V)\right)$.

Therefore, the output of our numerical algorithm would be summarized by correspondences $V_n^{h,\Sigma}$ under the action of a globally convergent operator $B^{h,\Sigma}$. From the application of operator $B^{h,\Sigma}$ on $V_n^{h,\Sigma}$, we can choose an approximate policy function $y = g_n^{y,h,\Sigma}(x,z,v)$, and a transition function $v_+(z_+) = g_n^{v,h,\Sigma}(x,z,v;z_+)$. From the computed selections we can generate approximate SCE paths $\{x_t(z^t), y_t(z^t)\}_{t=0}^\infty$. Sections 5 and 6 illustrate examples of such operators, and their application to different dynamic models.

4.2 Convergence and Accuracy Properties

We finally show that our discretized operator $B^{h,\Sigma}$ has good convergence properties: The fixed point of this operator $V^{*,h,\Sigma}$ contains equilibrium correspondence V^*, and it converges uniformly to this limit point as we refine the approximations. The proof of this result extends the convergence arguments of Beer (1980) to a dynamic setting.

Theorem 4.1 *For given h, Σ, let $V_0 \supseteq V^*$. Let $V_n^{h,\Sigma} = B^{h,\Sigma}(V_{n-1}^{h,\Sigma})$ for all $n \geq 1$. Then, (i) $V_n^{h,\Sigma} \supseteq V^*$ for all n; (ii) $V_n^{h,\Sigma} \to V^{*,h,\Sigma}$ as $n \to \infty$; and (iii) $V^{*,h,\Sigma} \to V^*$ as $h \to 0$ and $\Sigma \to \infty$.*

As stated in the theorem, three points are to be emphasized from these results. First, the set of numerical solutions always contains the equilibrium correspondence. Second, the iteration process is globally convergent. And third, as we refine these approximations, the fixed point of our numerical algorithm shrinks to the equilibrium correspondence.

We now establish uniform convergence over accuracy parameters (h, Σ). Hence, the approximation error is directly correlated with the mesh size of the discretizations. For correspondences $V_n^{h,\Sigma}$ and V, consider the distance $d(graph(V_n^{h,\Sigma}), graph(V))$, where d refers to the Hausdorff metric.

Theorem 4.2 *Under the conditions of Theorem 4.1, for any given $\epsilon > 0$ there are $\hat{\Sigma}, \hat{h}, \hat{n}$ such that the distance $d(graph(V_n^{h,\Sigma}), graph(V^*)) \leq \epsilon$ for all $\Sigma \geq \hat{\Sigma}, h \leq \hat{h}, n \geq \hat{n}$.*

Hence, for any sufficiently close discretization (Σ, h, n), all approximate solutions (x, z, v) are within an ϵ-ball of $graph(V^*)$; further, an ϵ-ball of $graph(V_n^{h,\Sigma})$ contains $graph(V^*)$. This important approximation result comes directly from the construction of our numerical operator $B^{h,\Sigma}$ that preserves equilibrium solutions and compactness over the iteration process. As already remarked, if the equilibrium correspondence V^* is just a function, then Theorem 4.2 implies the existence of error bounds for the approximate solutions. Indeed, these bounds follow directly from the size of the errors of the discretization procedure under parameters (h, Σ).

5 Stochastic OLG Economies

OLG models have become quite relevant in the analysis of several macro issues, such as the funding of social security, the optimal profile of savings and investment over the life cycle, the effects of various fiscal and monetary policies, and the evolution of future interest

rates and asset prices under current demographic trends.[4] As already stressed, there are no known convergent procedures for the computation of sequential competitive equilibria in OLG models even for frictionless economies with complete financial markets. Our approach delivers a reliable, computable algorithm for the solution of competitive equilibria in a general class of OLG models. As shown below, the application of standard numerical methods that build on the existence of a continuous policy function is not adequate for the computation of these economies. Indeed, a continuous Markov equilibrium may not exist – or there could be a vast multiplicity of equilibria. Citanna and Siconolfi (2010) establish *generic* existence of this Markovian property of equilibrium under the additional assumption that the number of agents is sufficiently large. Of course, for computational reasons many economies of practical interest contain a limited number of agents which are given as model primitives; further, this recursive representation is not necessarily continuous.

5.1 The Economic Environment

At each date a new generation made up of 2 agents appears in the economy. Each agent is alive for 2 periods. Let $\left(i, z^t\right)$ denote an agent of type $i = 1, 2$ born at date-event $z^t = (z_0, z_1, \cdots, z_t)$. There are 2 perishable commodities available for consumption at any given date-event. Let good 1 be the numeraire commodity, and p the relative price of good 2. There are two assets. The first asset is a one-period risk-free bond trading at price $q^b(z^t)$. The second is a Lucas tree, trading at price $q^s(z^t)$. The tree generates a random stream of dividends $d(z_t)$. Let $(\theta^{b,i,z^t}, \theta^{s,i,z^t})$ be a pair of bond and share holdings of agent $\left(i, z^t\right)$. Shares cannot be sold short: $\theta^{s,i,z^t} \geq 0$. Each individual faces the following budget constraint:

$$p(z^t) \cdot c^{i,z^t}(z^t) + \theta^{b,i,z^t}(z^t) q^b(z^t) + \theta^{s,i,z^t}(z^t) q^s(z^t) \leq p(z^t) \cdot e^{i,z^t}(z_t) \tag{7}$$

[4]For instance, see Conesa, Krueger and Kitao (2009), Geanakoplos, Magill and Quinzii (2003), Gourinchas and Parker (2002), Imrohoroglu, Imrohoroglu, and Joines (1995), Storelesletten, Telmer and Yaron (2004), and Ventura (1999).

$$p(z^{t+1})\cdot c^{i,z^t}(z^{t+1}) \leq \theta^{b,i,z^t}(z^t)+\theta^{s,i,z^t}(z^t)[d(z_{t+1})+q^s(z^{t+1})]+p(z^{t+1})\cdot e^{i,z^t}(z^{t+1}) \text{ all } z^{t+1}|z^t. \tag{8}$$

The utility function U^i is separable over consumption of different dates:

$$U^i\left(c^{i,z^t};z^t,z^{t+1}\right)=u^i\left(c^{i,z^t}\left(z^t\right),z_t\right)+\beta\sum_{z^{t+1}\in\mathbf{Z}}v^i\left(c^{i,z^t}\left(z^{t+1}\right),z_{t+1}\right)\pi\left(z^{t+1}|z^t\right). \tag{9}$$

Assumption 5.1 *For each $z \in Z$ the one-period utility functions $u^i(\cdot,z), v^i(\cdot,z) : \mathbb{R}_+ \to \mathbb{R}\cup\{-\infty\}$ are increasing, strictly concave, and continuous. These functions are also continuously differentiable at every interior point $c > 0$.*

As before, a SCE is a collection of vectors $\left\{\left(c^{i,z^t}\left(z^t\right),c^{i,z^t}\left(z^{t+1}\right),\theta^{i,z^t}\left(z^t\right)\right)_{i=1}^{I},p(z^t),q\left(z^t\right)\right\}_{t\geq 0}$ such that each consumption-savings plan $\left\{c^{i,z^t}(z^t),c^{i,z^t}(z^{t+1}),\theta^{i,z^t}(z^t)\right\}$ solves the constrained-utility maximization of the agent, and goods and assets markets clear.

Note that in this economy the aggregate commodity endowment is bounded by a portfolio-trading plan [Santos and Woodford (1997)], and hence asset pricing bubbles cannot exist in a SCE. Therefore, equilibrium asset prices must be bounded at each date. It follows that the existence of a SCE can be established by standard methods [e.g., Balasko and Shell (1980), and Schmachtenberg (1988)].

5.2 Lack of Recursive Equilibria on the Natural State Space

Let us first discuss the model specification of Kubler and Polemarchakis (2004) where the real asset is not available. The intertemporal objective of agent of type 1 is given by

$$-\frac{1024}{\left(c_1^{1,z^t}\right)^4}+E_{z_{t+1}|z^t}\left[-\frac{1024}{\left(c_1^{1,z^t}(z_{t+1})\right)^4}-\frac{1}{\left(c_2^{1,z^t}(z_{t+1})\right)^4}\right]$$

while that of agent of type 2 is given by

$$-\frac{1}{\left(c_1^{2,z^t}\right)^4}+E_{z_{t+1}|z^t}\left[-\frac{1}{\left(c_1^{2,z^t}(z_{t+1})\right)^4}-\frac{1024}{\left(c_2^{2,z^t}(z_{t+1})\right)^4}\right].$$

Each individual receives a random endowment of good 1 in their first period of life. Specifically, $e_1^{1,z^t}(z_t) = 10.4$, $e_1^{2,z^t}(z_t) = 2.6$ if $z_t = z_1$, and $e_1^{1,z^t}(z_t) = 8.6313$, $e_1^{2,z^t}(z_t) = 4.3687$ if $z_t = z_2$. Endowments during the second period of life are deterministic and include positive amounts of both goods. Namely, $e^{1,z^t}\left(z^{t+1}\right) = (12, 1)$ and $e^{2,z^t}\left(z^{t+1}\right) = (1, 12)$.

Kubler and Polemarchakis show that bond holdings turn out to be equal to zero in the two states. To determine consumption when old we must know the realization of the endowment when young.[5] Bond holdings and current shocks are not enough to pin down the dynamic behavior of equilibrium. In other words, the model does not admit a Markov equilibrium representation over the natural state space. The specific configuration of equilibrium is as follows: At any state history z^{t-1} with $z_{t-1} = z_1$, and for any possible value of the shock today $\left(c_1^{1,z^{t-1}}\left(z^t\right), c_2^{1,z^{t-1}}\left(z^t\right)\right) = (10.4, 2.6)$, $\left(c_1^{2,z^{t-1}}\left(z^t\right), c_2^{2,z^{t-1}}\left(z^t\right)\right) = (2.6, 10.4)$, and $q = 1, p = 1$. Likewise, for any state history z^{t-1} with $z_{t-1} = z_2$, and for any possible value of the shock today $\left(c_1^{1,z^{t-1}}\left(z^t\right), c_2^{1,z^{t-1}}\left(z^t\right)\right) = (8.4, 1.4)$, $\left(c_1^{2,z^{t-1}}\left(z^t\right), c_2^{2,z^{t-1}}\left(z^t\right)\right) = (4.6, 11.6)$, and $q = 1, p = 7.9$.

What would happen if we approximate this economy by standard methods? To answer this question we applied a projection method with collocation and piecewise linear interpolation. This collocation method approximates the Euler equation to search for a continuous equilibrium function over the natural state space. The computed equilibrium function delivers reasonable Euler equation residuals (i.e., of the order of 10^{-5}) and a researcher may be led to believe that this function is a good approximate solution; however, the computed prices and allocations are quite different from those of the exact equilibrium; see Table 1.

Statistics	q	$c_1^{1,z^{t-1}}$	$c_2^{1,z^{t-1}}$	$c_1^{2,z^{t-1}}$	$c_2^{2,z^{t-1}}$
$(\mu_{true}, \mu_{projection})$	(1.0,0.6)	(9.7,9.7)	(2.0,1.7)	(3.6,3.8)	(11.0,11.3)
$(\sigma^2_{true}, \sigma^2_{projection})$	(0.0,0.05)	(1.0,0.2)	(0.36,0.81)	(1.0,0.09)	(0.36,0.08)

Table 1: Statistical properties of the true equilibrium vs. an equilibrium generated by the projection method. Statistics: Mean μ and variance σ^2.

[5]Because of an indeterminacy problem of the Euler equation, we can approximate the equilibrium of this more limited economy by letting the stock of trees go to zero.

The relative price of good 2 is a function of the endowment in the previous period. The price is not signaled by the natural state space as there is no trade among generations. The equilibrium relative price of good 2 can take on two values and asset holdings take on one single value. This observation may explain the large differences in Table 1 between the simulated moments generated by the true and computed solutions. Indeed, the computed function by the projection method takes on a single value for the relative price of good 2 midway between the two possible equilibrium values.

5.3 The Recursive Equilibrium Algorithm

A recursive representation of equilibria can be readily recovered on an enlarged state space composed of the natural state variables and the shadow values of investment as auxiliary variables. For the present economy of Kubler and Polemarchakis (2004), the Markov equilibrium correspondence can be defined as follows:

$$V^*(\theta_0, z_0) = \left\{ \begin{array}{c} \left(D_c v^1(c^{1,z^0}(z_0), z_0), D_c v^2(c^{2,z^0}(z_0), z_0)\right) : \\ \left\{\left(c^{i,z^t}(z^t), c^{i,z^t}(z^{t+1}|z^t), \theta^{i,z^t}(z^{t+1})\right)_{i=1}^{2}, p(z^t), q(z^t)\right\}_{t\geq 0} \text{ is a SCE} \end{array} \right\}. \tag{10}$$

Operator B will build on the first-order and market-clearing conditions. After some algebra, these conditions can be written as:

$$\frac{12+p+\theta}{4p^{1/5}+p} + \frac{4+48p-4\theta}{p^{1/5}+4p} = 13 \tag{11}$$

$$\frac{q}{\left(e_1^1(z)-q\theta\right)^5} = \frac{\pi[z_+=z_1|z]}{\left(\frac{12+p(z_+)+\theta_+}{1+0.25p(z_+)^{4/5}}\right)^5} + \frac{1-\pi[z_+=z_1|z]}{\left(\frac{12+p(z_+)+\theta_+}{1+0.25p(z_+)^{4/5}}\right)^5} \tag{12}$$

$$\frac{q}{\left(e_1^2(z)+q\theta\right)^5} = \frac{\pi[z_+=z_1|z]}{\left(\frac{1+12p(z_+)-\theta_+}{1+4p(z_+)^{4/5}}\right)^5} + \frac{1-\pi[z_+=z_1|z]}{\left(\frac{1+12p(z_+)-\theta_+}{1+4p(z_+)^{4/5}}\right)^5}. \tag{13}$$

Then, for each given (z,θ), and $m \in V(z,\theta)$ we have that $m \in BV(z,\theta)$ if there are

$(q, p, z_+, \theta_+, m(z_+))$ such that

$$\frac{12+p+\theta}{4p^{1/5}+p}+\frac{4+48p-4\theta}{p^{1/5}+4p}=13 \tag{14}$$

$$\frac{q}{\left(e_1^1(z)-q\theta\right)^5}=Em_+^1 \tag{15}$$

$$\frac{q}{\left(e_1^2(z)+q\theta\right)^5}=Em_+^2. \tag{16}$$

The numerical implementation of our recursive equilibrium algorithm is quite simple in this model. The only equilibrium portfolio is $\theta = 0$. However, to test the algorithm we consider a slightly larger domain $[\underline{\theta}, \overline{\theta}]$, with $\underline{\theta} < 0 < \overline{\theta}$. The family of simplices we employ is given by the set of N intervals of the form $[\underline{\theta}+nh, \underline{\theta}+(n+1)h]$, for $n = 1, 2, ..., N$, and h is such that $\overline{\theta} = \underline{\theta}+(N+1)h$. The only equilibrium price for the bond is $q = 1$. This value together with the definition of the shadow values of investment are now used to set up our discretization for the initial step correspondence. Let $\theta \in [\underline{\theta}+nh, \underline{\theta}+(n+1)h]$. Then

$$V_0^{h,\Sigma}(\theta, z) = \bigcup_{i,j}\{(m^1, m^2) \in \left[\tfrac{1}{e^1(z)-(\underline{\theta}+nh)-i\Sigma}, \tfrac{1}{e^1(z)-(\underline{\theta}+nh)-(i+1)\Sigma}\right] \times \left[\tfrac{1}{e^2(z)+(\underline{\theta}+(n+1)h)-j\Sigma}, \tfrac{1}{e^2(z)+(\underline{\theta}+(n+1)h)-(j+1)\Sigma}\right]\}$$

for $i = 1, ..., N_i, j = 1, ..., N_j$, and $e^1(z) - (\underline{\theta}+nh) - (N_i+1)\Sigma = (\underline{\theta}+(n+1)h)$, and $e^2(z) + (\underline{\theta}+(n+1)h) - (N_j+1)\Sigma = e^2(z) + (\underline{\theta}+nh)$. This specification is also very convenient because we have partitioned the image of the correspondence into $N_i \times N_j$ pieces at each element of the simplex of the domain of asset holdings. Iteration of operator $B^{h,\Sigma}$ will eliminate those pieces that cannot be linked to a continuation value. After k iterations, correspondence $V_k^{h,\Sigma}$ is conformed by the union of those pieces that have not been eliminated. Operator $B^{h,\Sigma}$ is then defined as follows: For any given simplex, an element (i, j) of $V_k^{h,\Sigma}$ remains in $B^{h,\Sigma}V_k^{h,\Sigma} = V_{k+1}^{h,\Sigma}$ if there is at least one $\theta \in [\underline{\theta}+nh, \underline{\theta}+(n+1)h]$, and a pair $(m_1, m_2) \in \left[\tfrac{1}{e^1(z)-(\underline{\theta}+nh)-i\Sigma}, \tfrac{1}{e^1(z)-(\underline{\theta}+nh)-(i+1)\Sigma}\right] \times \left[\tfrac{1}{e^2(z)+(\underline{\theta}+(n+1)h)-j\Sigma}, \tfrac{1}{e^2(z)+(\underline{\theta}+(n+1)h)-(j+1)\Sigma}\right]$ for which we can find $(q, p, z_+, \theta_+, m(z_+))$ satisfying conditions (14-16).

6 International Risk Sharing

A growing literature has developed to explore the performance of business cycle models under market imperfections leading to limited risk sharing. As documented in various papers [e.g., Backus, Kehoe and Kydland (1992)], standard versions of the neoclassical growth model cannot account for certain co-movements of macroeconomic aggregates. We now show that our reliable algorithm can naturally be applied to the computation of two-country models with real and financial frictions.

6.1 The Economic Environment

We just outline an economy in the spirit of Kehoe and Perri (2002) in which we include shocks on preferences and taxes. There are two-countries with a representative household in each country. Each economy is affected by a vector of shocks z that follow a Markov chain. There is a unique aggregate good. Production technologies are country specific. Labor and capital stocks cannot be moved across countries, but limited international borrowing is possible. Assets include physical capital and bonds.

The representative household of country $i = 1, 2$ has preferences over stochastic sequences of consumption and labor given by the utility function

$$E\left[\sum_{t=0}^{\infty} \beta^t u^i\left(c_t^i, l_t^i, z_t\right)\right]. \tag{17}$$

Function $u^i(\cdot, \cdot, z_t) : \mathbb{R}^2 \to \mathbb{R}$ is increasing in $c^i \geq 0$ and decreasing in $l^i \in [0, 1]$, strictly concave, and twice continuously differentiable. Stochastic consumption plans $\left(c_t^i\right)_{t\geq 0}$ are financed by commodity endowments, after-tax capital returns, labor income, and lump-sum transfers. These values are expressed in terms of the single good, which is taken as the numeraire commodity of the system at each date-event, z^t. For a given rental rate r_t^i and wage w_t^i in country i, the representative household offers $k_t^i(z^{t-1}) \geq 0$ units of capital accumulated from the previous period, and supplies $l_t^i(z^t)$ units of labor.

One-period bonds can be traded at all times. Let $b^i(z^t, \xi_{t+1}^l(z^t))$ denote bond holdings

of agent i, where $\xi_{t+1}^l(z^t)$ is a representative element of a given partition of the possible successors $z^{t+1}|z^t$. Hence, $\cup_l \xi_{t+1}^l(z^t)$ equals the set of all $z^{t+1}|z^t$, and $\xi_{t+1}^{l'}(z^t) \cap \xi_{t+1}^l(z^t) = 0$ whenever $l' \neq l$. A bond is a promise to deliver 1 unit of the consumption good whenever $z^{t+1} \in \xi_{t+1}^l(z^t)$, and 0 otherwise. This specification allows for a full set of state contingent bonds if $\xi_{t+1}^l(z^t)$ is a unique element for each l. An uncontingent bond pays one unit of the good for any possible future state. Let $q(z^t, \xi_{t+1}^l(z^t))$ be the price of a bond issued at z^t.

The representative household of country i is subject to the following sequence of budget constraints:

$$c_t^i\left(z^t\right) + k_{t+1}^i\left(z^t\right) + b^i(z^t, \xi_{t+1}^l(z^t))q(z^t, \xi_{t+1}^l(z^t)) = w_t^i\left(z^t\right) l_t^i\left(z^t\right) + \tag{18}$$
$$(1 - \tau_k^i(K^i))r_t^i\left(z^t\right) k_t^i\left(z^{t-1}\right) + (1-\delta)\, k_t^i\left(z^{t-1}\right) + e_t^i\left(z_t\right) + b^i(z^{t-1}, \xi_t^l(z^{t-1})) + T_t^i\left(z^t\right),$$
$$\text{for all } z^t,\ t \geq 0,\ \text{given } k_0^i.$$

Endowments $e_t^i(z^t)$ are strictly positive and depend only on the current realization of the shock z_t. Capital income is taxed according to function τ_k, which may depend on the aggregate capital stock, K_t^i, or some other state variables. This tax function is assumed to be positive, continuous, and bounded away from 1. Tax revenues are rebated back to the representative consumer as lump-sum transfers $T_t^i\left(z^t\right) = \tau_k^i(K^i) r_t^i\left(z^t\right) K_t^i\left(z^t\right)$.

As in Kehoe and Perri (2002), we consider two scenarios for financial markets. A *debt-constrained* economy and a *liquidity-constrained* economy. In the debt-constrained economy, consumers have a complete menu of contingent bonds. Financial markets would be therefore complete, except for the fact that there are constraints on debt holdings. Debt repudiation is possible and entails permanent exclusion from financial markets. As a result, holdings of debt are constrained by the following individually rational constraint at every possible node z^t:

$$E_{z^t} \sum_{\tau=t}^{\infty} \left(\beta^i\right)^\tau u^i\left(c_\tau^i, l_\tau^i, z_\tau\right) \geq V^{i,aut}(K_{t-1}^i(z^t), z^t), \text{ for all } t \geq 0. \tag{19}$$

Here, $V^{i,aut}$ is the expected discounted utility value for autarky as a result of zero bond trading for country i at all dates after z^t. Hence, $K^i_{t-1}(z^t)$ is the average level of physical capital of country i starting at node z^t. It is important to stress that the representative agent in each country makes choices on their capital holdings, k^i_t, assuming that the average value of the stock of capital K^i_{t-1} is given. As is typical in many models with externalities, no individual agent realizes that her choices affect the aggregate borrowing constraint (19). Therefore, in this setting the constraint set is convex, and so the first-order approach can be used to characterize equilibria.

In the liquidity-constrained economy, households can trade quantities $b^i(z^t)$ of a single uncontingent bond that yields one unit of the commodity for all states, subject to the following exogenous constraint:

$$b^i(z^t) \geq -\Omega^i, \tag{20}$$

where Ω^i is some positive number.

Because of constant-returns-to-scale technologies, we can focus on the problem of a representative firm without loss of generality. After observing the current shock z the firm rents K^i units of capital and hires L^i units of labor. The total quantity produced of the single aggregate good is given by a production function $A^i_t F\left(K^i_t, L^i_t\right)$, where A^i_t is a TFP index and $F\left(K^i_t, L^i_t\right)$ is the direct contribution of the firm's inputs to the production of the aggregate good. At every date-event z^t factors of production are demanded by the firm to the point in which the marginal productivity of capital equals the rental rate r^i_t and the marginal productivity of labor equals the wage w^i_t. We shall maintain the following standard conditions on production function F. Let $D_1F\left(K, L\right)$ be the derivative of F with respect to K.

Assumption 6.1 *$F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, concave, continuous, and linearly homogeneous. This function is continuously differentiable at each interior point (K, L); moreover, $\lim_{K\to\infty} D_1F\left(K, L\right) = 0$ for all $L > 0$.*

6.2 Competitive Equilibrium

Definition 6.1 *A SCE is a tax function $\tau_k^i(K)$, and a collection of vectors* $\left(\{c_t^i(z^t), l_t^i(z^t), k_{t+1}^i(z^t), b^i(z^t, \xi_{t+1}^l(z^t)), K_{t+1}^i(z^t), L_t^i(z^t), r_t^i(z^t), w_t^i(z^t)\}_{i=1,2}, q(z^t, \xi_{t+1}^l(z^t))\right)_{t\geq 0}$ *that satisfy the following conditions:*

(i) Constrained-utility maximization: For $i = 1, 2$ the sequence $\{c_t^i, l_t^i, k_{t+1}^i, b_t^i\}_{t\geq 0}$ solves the maximization problem for the objective (17) subject to the sequence of budget constraints (18), as well as constraint (19) for the debt-constrained economy, and constraint (20) for the liquidity-constrained economy.

(ii) Market clearing in all the markets: Goods, capital, labor, and bond markets.

We are just extending the definition of SCE of Kehoe and Perri (2002) with the addition of taxes. Note that in this economy international borrowing allows for imports of the aggregate good produced abroad – available for consumption and investment – but the representative firm can only hire local inputs – capital and labor.

There does not seem to be a general proof of existence of competitive equilibria for infinite-horizon economies with distortions. We are aware of a related contribution by Jones and Manuelli (1999), but their analysis is not directly applicable to models with incomplete markets or externalities. Hence, the Appendix outlines a proof of the following result.

Proposition 6.2 *A SCE exists.*

6.3 Bounds on Equilibrium Allocations and Prices

The Appendix shows existence of positive constants K^{max} and K^{min} such that for every equilibrium sequence of physical capital vectors $\{k_{t+1}^i(z^t))\}_{t\geq 0}$ if $K^{max} \geq \sum_{i=1}^2 k_0^i(z^0) \geq K^{min}$ then $K^{max} \geq \sum_{i=1}^2 k_{t+1}^i(z^t) \geq K^{min}$ for all z^t. Hence, in what follows the domain of aggregate capital will be restricted to the interval $[K^{min}, K^{max}]$. We also show that every equilibrium sequence of factor prices $\{r_t^i(z^t), w_t^i(z^t)\}_{t\geq 0}$ is bounded.

To build operator B, we need to bound the equilibrium shadow values of investment. For this purpose, we introduce the following dynamic programming argument: We define an auxiliary value function of an individual sequential optimization problem. For a given sequence of factor and bond prices and aggregate capital $(\mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0)) = \{r_t(z^t), w_t(z^t), q_t\left(z^t\right), K_{t+1}\left(z^t\right)\}_{t\geq 0}$, let

$$J^i(k_0^i, b_0^i, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0)) = \max E \sum_{t=0}^{\infty} \beta^t u^i(c_t(z^t), l_t(z^t), z_t)$$

subject to the sequence of budget constraints (18), as well as constraint (19) for the debt-constrained economy, and constraint (20) for the liquidity-constrained economy, for given initial conditions k_0^i, b_0^i. That is, $J^i(k_0^i, b_0^i, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$ is the maximum utility attained for initial k_0^i, b_0^i, over an expected future sequence of equilibrium prices and tax rebates.

For every bounded sequence $(\mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$, the value function $J^i(k_0^i, z_0, b_0^i, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$ is well defined, and continuous. Moreover, mapping $J^i(\cdot, \cdot, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$ is increasing, concave, and differentiable with respect to k_0^i and b_0^i [cf. Rincon-Zapatero and Santos (2009)]. Let $D_{k,b}J^i(\cdot, \cdot, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$ be the partial derivative of function $J^i(\cdot, \cdot, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$ with respect to (k_0, b_0). Then, $D_{k,b}J^i(\cdot, \cdot, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$ varies continuously with $(k_0^i, b_0^i, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$. The next result readily follows from these regularity properties of the value function.

Proposition 6.3 *For all SCE* $\left(\{c_t^i(z^t), l_t^i(z^t), k_{t+1}^i(z^t), b^i(z^t, \xi_{t+1}^l(z^t)), K_{t+1}^i(z^t), L_t^i(z^t), r_t^i(z^t), w_t^i(z^t)\}_{i=1,2,}\right)_{t\geq 0}$ *with* $K^{max} \geq \sum_{i=1}^2 k_0^i(z^0) \geq K^{min}$, *there is a constant vector* $\widehat{\gamma} = (\gamma, \gamma)$ *for* $\gamma > 0$ *such that* $0 \leq D_{k,b}J^i(\cdot, \cdot, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0)) \leq \widehat{\gamma}$ *for all* z^t.

Observe that these bounds apply to the following types of utility functions: (i) Both function $u(\cdot,\cdot,z)$ and its derivative are bounded; (ii) function $u(\cdot,\cdot,z)$ is bounded, and its derivative function is unbounded; and (iii) both function $u(\cdot,\cdot,z)$ and its derivative are unbounded. Phelan and Stacchetti (2001) deal with case (i) and Kubler and Schmedders (2003) consider utility functions of type (iii). We provide a uniform method of proof that covers all three cases, as well as production functions with bounded and unbounded derivatives, and exogenous and endogenous constraints. As a matter of fact, Proposition 6.3 fills an important gap in the literature for production economies with heterogeneous consumers and market frictions, since no general results are available on upper and lower bounds for equilibrium allocations and prices.

For any initial distribution of capital $k_0 = (k_0^1, k_0^2)$, bonds $b_0 = (b_0^1, b_0^2)$ and a given shock z_0, the shadow values of investment that belong to the equilibrium correspondence are defined as

$$V^*(k_0, b_0, z_0) = \left\{ \begin{array}{c} \{D_{k,b} J^i(k_0^i, b_0^i, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))\}_{i=1,2} : \\ \text{There is a SCE} \end{array} \right\}. \tag{21}$$

Hence, the set $V^*(k_0, b_0, z_0)$ contains all current equilibrium shadow values of investment returns m_0^i, for every household i.

Corollary 6.4 *Correspondence V^* is non-empty, compact-valued, and upper semicontinuous.*

This corollary is a straightforward consequence of Propositions 6.2 and 6.3. These bounds insure that our operator B maps compact sets into compact sets [cf., Assumption 3.1]. The construction of B follows the same steps of the preceding section.

6.4 The Recursive Equilibrium Algorithm

The natural state space is conformed by the space of shocks and the distribution of wealth (namely, individual country holdings of the capital stock and bonds). Because of financial and real frictions, auxiliary variables are also needed to guarantee a recursive representation

of equilibria. For the economy with exogenous debt limits we enlarge the state space with the shadow values of investment. For the economy with endogenous debt limits we enlarge the state space with both the shadow values of investment, m, and values of participation, p.

Note that in equilibrium $b^1(z^t, \xi^l_{t+1}(z^t)) = -b^2(z^t, \xi^l_{t+1}(z^t))$. Hence, in the sequel we let b be the share holdings of country 1. Then, the equilibrium correspondence $V^*(b, k^1, k^2, z^1, z^2)$ is a map from the space of possible values for each country's capital stock and shocks, and bond holdings for agent 1, into the set of possible equilibrium values for the auxiliary variables.

For the economy with exogenous constraints, both b, q are scalars, and the shadow values of investment are defined as:

$$m^i(b, k^1, k^2, z^1, z^2) = u^i_c \left(A^i(z^i) F_k(k^i, l^i) + (1 - \delta) k^i \right), \tag{22}$$

We can now build operator B from the first-order and market-clearing conditions. For any pair of equilibrium values for the shadow values of investment $(m^1, m^2) \in V^*(b, k^1, k^2, z^1, z^2)$, there must be bond prices q, multipliers λ, tomorrow's bond holdings, b_+, capitals, k^1_+, k^2_+, and shadow values of investment $(m^1_+, m^2_+) \in V^*(b_+, k^1_+, k^2_+, z^1_+, z^2_+)$ such that the short-run equilibrium conditions

$$u^i_c = \lambda^i + \beta^i E m^i_+ \tag{23}$$

$$u^i_c A^i F_L = u^i_l \tag{24}$$

are satisfied. Here $\lambda^i \geq 0$, with strict inequality only if today's borrowing constraint binds. As before, E is the expectations operator.

Analogously, for the economy with endogenous constraints, given a tuple of equilibrium shadow values of investment and participation, $(m^1, m^2, p^1, p^2) \in V^*(b, k^1, k^2, z^1, z^2)$, it must be possible to find continuation values that satisfy the following short-run equilibrium

conditions:

$$u_c^i = \zeta^i \beta^i E m_+^i \tag{25}$$

$$u_c^i A^i F_L = u_l^i \tag{26}$$

$$p^i = u + \beta^i E p_{+.}^i \tag{27}$$

In the Euler equation above, $\zeta^i \geq 1$ is a ratio of multipliers corresponding to the participation constraints. Therefore, $\zeta^i > 1$ only if tomorrow's participation constraint is binding.

As before, we start with a correspondence $V_0 \supseteq V^*$. It is easy to bound this initial candidate V_0, since the lowest value of endowments are a lower bound for consumption, and the marginal utility of consumption can be used to bound asset prices as discounted values of dividends. It is also straightforward to derive bounds for the value of participation p_0.

For the purposes of presentation, let us just deal with the scenario of the exogenous borrowing constraint (20) where values of participation are not required. For given (b, k^1, k^2, z^1, z^2), operator B dictates that $(m^1, m^2) \in BV_n(b, k^1, k^2, z^1, z^2)$ if we can find continuation shadow values of investment $(m_+^1, m_+^2) \in V_n(b_+, k_+^1, k_+^2, z_+^1, z_+^2)$, a bond price q, and multipliers (λ^1, λ^2), such that optimality conditions (23-24) are satisfied. If we cannot find continuation values that satisfy the previous conditions, then $(m^1, m^2) \notin BV_n(b, k^1, k^2, z^1, z^2)$. A new correspondence $V_{n+1} = B(V_n)$ is defined after proceeding with these computations over every possible value (b, k^1, k^2, z^1, z^2).

Iterating over operator B we get new candidate values for the shadow values of investment and values for participation over the short-run equilibrium conditions (25-27). Our algorithm can then be used to generate a sequence of approximations to the equilibrium correspondence via the recursion $V_{n+1} = B(V_n)$.

For the numerical implementation of the algorithm, we assume pre-specified intervals for the values of bond and capital holdings. We then partition the state space over a set of vertex points with grid size h. The step correspondence approximating V_0 over each

element in the partition of the state $s_{ijl} \equiv [b_i, b_{i+1}] \times [k^1_j, k^1_{j+1}] \times [k^2_l, k^2_{l+1}]$ can be defined as

$$V^h_0(b, k^1, k^2, z^1, z^2) = \cup_{(b,k^1,k^2,z^1,z^2)\in s_{ijl}} V_0(b, k^1, k^2, z^1, z^2)$$
$$V^h_0(b, k^1, k^2, z^1, z^2) = \cup_{(b,k^1,k^2,z^1,z^2)\in s_{ijl}} V_0(b, k^1, k^2, z^1, z^2).$$

The image of this correspondence comprises the shadow values of investment (m^1, m^2). Hence, a simple outer approximation $\mathcal{C}^\Sigma\left(B^h(V)\right)$ would be a finite collection of hypercubes for vectors (m^1, m^2). This completes the numerical implementation of operator $B^{h,\Sigma}$, defined over computable step correspondences.

We use our method to compute SCE of this two-country model with endogenous and exogenous borrowing constraints. In both scenarios we find that the equilibrium correspondence converges to a function (up to numerical accuracy of 10^{-6}), which indicates that the SCE is unique for given initial conditions. This is the only model of the paper where computational time is a substantial issue. The basic form of our algorithm is fairly easy to implement: It only requires to search for continuation values over the short-run equilibrium conditions required by operator B are satisfied. As this process of search is independent across states, it is straightforward to use parallel computing. In terms of running times, as in most algorithms the choice of initial guess matters greatly. The initial guess we employed was the solution of a similar economy but with complete markets and no distortions, which can easily be secured with a standard dynamic programming algorithm. Our grid considers 51 equally spaced points for K and 501 points for m for each country $i = 1, 2$. We ran our C++ MPI code using an IBM Server 1350 Cluster, with 50 Xeon 2.3GHZ processors. The average time per iteration of operator B was 24 minutes. The program took 94 iterations to converge to a desired level of accuracy. These times were lower in the liquidity-constrained economy.

6.5 Quantitative Experiments

We now explore the quantitative implications of the above two financial scenarios. For comparison purposes we will also report results for the model with complete markets to be solved under standard dynamic programming techniques.

We assume a one-period utility with stochastic shock $\nu^i(z)$:

$$u^i(c,l,z) = \nu^i(z)\frac{\left[c^\eta(1-l)^{1-\eta}\right]^{1-\sigma}}{1-\sigma}, \tag{28}$$

and a Cobb-Douglas production function:

$$AF(K,L) = AK^\alpha(L)^{1-\alpha}. \tag{29}$$

We shall use the following standard parameter values: $\alpha = 0.36, \eta = 0.36$, and $\sigma = 2$. From quarterly data, we let $\beta = 0.99$ and $\delta = 0.025$. We consider a discrete VAR process for the technology shocks with four possible states: $(A^1 = 0.95613, A^2 = 0.95613)$, $(A^1 = 0.95613, A^2 = 1.04480)$, $(A^1 = 1.04480, A^2 = 0.95613)$, $(A^1 = 1.04480, A^2 = 1.04480)$. These states evolve according to the transition matrix

$$\pi = \begin{bmatrix} 0.83022 & 0.07849 & 0.07803 & 0.01326 \\ 0.10821 & 0.77567 & 0.00865 & 0.10747 \\ 0.10971 & 0.00793 & 0.77629 & 0.10607 \\ 0.01354 & 0.07934 & 0.07960 & 0.82752 \end{bmatrix}.$$

Table 2 reports the simulated moments for the complete-markets economy, the debt-constrained economy, and the liquidity-constrained economy in which the borrowing limit $\Omega^i = 0$. The resulting simulated sample moments are in line with those reported in Kehoe and Perri (2002) who use a slightly different calibration and a different computational method.

Only the debt-constrained economy offers a chance of generating reasonable correlations. In the first three scenarios, preferences are non-stochastic ($\nu(z) = 1$), and there are no taxes ($\tau = 0$). The last column of Table 2 reports a slightly different experiment for the liquidity-constrained economy with stochastic preferences and taxes. The idea is to see how

shocks on preferences and taxes may improve the performance of the liquidity-constrained economy. We assume that $\nu^i = 1.05$ if $A^i > 1$, and $\nu^i = 0.95$ if $A^i \leq 1$. Hence, the representative household is more optimistic (or more willing to consume) in the event of a good productivity shock. Also, $\tau^i = 0.30$ if $A^i > 1$, and $\tau^i = 0.25$ if $A^i \leq 1$. That is, taxes are also procyclical. With respect to the liquidity-constrained economy, this last calibration improves all the bilateral correlations; still, it does not do as well for the correlations of consumption c and GDP.

	Data	*complete markets*	*liquidity constrained*	*debt constrained*	*preferences/tax shocks*
Bilateral correlations					
c	0.32	0.8003	-0.8767	0.2264	-0.36
GDP	0.51	-0.5947	-0.7568	0.0170	-0.28
$Investment$	0.29	-0.9117	-0.9953	0.6037	0.41
labor	0.43	-0.9341	-0.8714	-0.1062	0.19

Table 2: Statistical properties of the economies with complete markets, and with exogenous or endogenous constraints.

In summary, in this section we apply our reliable algorithm to a two-country general equilibrium model with several real and financial frictions: Incomplete markets, exogenous and endogenous constraints, preference shocks, and taxes. We establish bounds for equilibrium allocations and prices as a key condition for the numerical implementation of our algorithm. Our model simulations broadly confirm the findings of Kehoe and Perri (2002): Endogenous debt constraints seem instrumental to fix some international business cycles anomalies. We here obtain a related result with procyclical preference shocks and taxation to improve the cross-country correlation of capital and labor. Our computational method can accommodate some other extensions (e.g., time-to-build, adjustment costs), or can be applied to related models of international investment [Bai and Zhang (2010)].

7 Concluding Remarks

This paper provides a theoretical framework for the numerical simulation of dynamic competitive-market economies in which the welfare theorems may fail to hold because of market frictions or the existence of an infinite number of generations. This includes various macroeconomic models with heterogeneous agents, incomplete financial markets, endogenous and exogenous borrowing constraints, taxes, and money. Equilibrium solutions are not amenable to computation using social planning problems because of the existence of real and financial frictions. They are not amenable to computation by projection methods with continuous equilibrium functions because a continuous recursive representation of equilibrium may not exist. And they are not amenable to computation by perturbation methods because the ergodic region may be quite large: Agents accumulate assets to accommodate idiosyncratic and aggregate risks. All these computational methods may actually generate large approximation errors.

To overcome these rather limiting technicalities, we propose a reliable *recursive equilibrium algorithm*. Our approach is intended to be quite general – available characterizations of equilibria are usually model-dependent. We consider an abstract framework that covers equilibrium models with various real and financial frictions, and resource and participation constraints. Convexity assumptions are not necessary, but certain mild continuity and time-separability conditions must be satisfied. That is, the model must be *recursive*: An equilibrium solution must be characterized by aggregate resource constraints and short-run optimality conditions comprising only variables of two contiguous time periods, t and $t+1$.

Under mild regularity conditions, we can define a non-empty Markov equilibrium correspondence that generates the set of all sequential competitive equilibria. This correspondence lies in an expanded state space, and can be obtained as the fixed point of an operator embedding all aggregate constraints and *short-run* equilibrium conditions. The iteration process under this operator is globally convergent for every initial guess containing the Markov equilibrium correspondence.

We provide a discretized version of this operator which is also globally convergent. This discretized operator iterates over correspondences rather than functions. As we refine the discretization process the fixed point of the discretized operator converges $uniformly$ to the Markov equilibrium correspondence on every compact subdomain. In the present general context, uniform convergence is a very strong approximation result. Actually, for economies with unique equilibria the nature of our approximation scheme makes it possible to derive uniform error bounds.

In the numerical implementation of the algorithm, the choice of auxiliary variables conforming the state space becomes critical. It is simplest to enlarge the state space with all endogenous and exogenous variables, but then the algorithm may not be computable. In the above applications, the computation of equilibria relied on first-order conditions in which the auxiliary variables were the shadow values of investment for each asset and for each agent. Under this choice of the state space the Euler equations were linear – speeding up the computation process. The linearity of the Euler equations was preserved in models with exogenous borrowing constraints. With endogenous borrowing constraints, continuation utility values were also added to the state space. The final objective is to work with a minimal extension of the state space that becomes operative at the computational stage.

Our quantitative analysis ends with the simulation of a stochastic overlapping generation economy and a business cycle model of international trading along the lines of Kehoe and Perri (2002). The overlapping generations economy was instrumental to illustrate some of the pitfalls that may occur in the computation of equilibrium solutions for non-optimal economies while using algorithms that search for a continuous equilibrium function over the natural space of state variables. These traditional algorithms cannot insure convergence of the approximate solution to the given equilibrium fixed point. As a matter of fact, the computed solution contained large approximation errors because of a poor choice of the state space.

In the numerical simulation of the two-country business cycle model, we contemplate various scenarios for cross-country risk sharing in a full-blown economic setting with capital accumulation, taxation, and preference shocks. Among all the financial arrangements, endogenous borrowing constraints improve substantially the predictions of the model relative to the data. This is in line with the findings of Kehoe and Perri (2002). As these authors point out, models with additional frictions may be necessary to make the theory fully compatible with the data. Under our recursive equilibrium algorithm, it was fairly easy to accommodate procyclical preference shocks and taxes. These extensions improve the cross-country correlation of investment and labor.

All of these results add to a large body of literature in macroeconomics and finance intended to overcome some severe limitations of the representative-agent paradigm. The quantitative analysis of nonoptimal dynamic economies is certainly a non-trivial task. Hence, reliable methods for the numerical approximation of these economies should prove very valuable. Feng (2011) generalizes our computational approach to quantify the welfare loss of time inconsistency in an economy with capital and labor taxation.

Of course, our methods must face some computational challenges. Iteration over correspondences is computationally much more costly than iteration over functions. The expansion of the state space along with iteration over sets should certainly be manifested into an additional computational burden. In the characterization of Markov equilibria, it is therefore imperative to select a set of auxiliary variables with a view towards minimizing the computational cost. The development of high-performance, parallel computing will certainly make our methods more attractive as the many computational tasks in our algorithm can be decentralized.

Finally, the numerical implementation of our algorithm starts with an initial correspondence of potential equilibrium values. In most numerical work it is necessary to bound the ergodic region. This task, however, may become much more delicate for nonoptimal economies since no general theory is available to bound asset prices and returns. In our

applications above we have developed various procedures to bound equilibrium allocations and prices by ruling asset pricing bubbles and by defining a value function for each household over future equilibrium paths. This value function is convenient because it can embed exogenous and endogenous borrowing constraints, as well as other real and financial frictions. Hence, market imperfections need not be explicitly considered to bound equilibrium allocations and prices. These techniques should certainly be valuable to establish feasible bounds in related models with heterogeneous agents and market distortions.

8 Appendix

In this Appendix we prove some key results formally stated in Sections 3 and 4. For convenience, we also offer a proof of existence for the model of Section 6, and establish equilibrium bounds. All other claims in the paper may rely on similar arguments.

Proof of Theorem 3.1: Let $\widehat{V}_0 \supset V^*$, and $\widehat{V}_i = B(\widehat{V}_{i-1})$ for all $i \geq 1$. To insure monotone convergence, let us now redefine these sets as $V_n = \cup_{i=n}^{\infty} \widehat{V}_i$, for all $n \geq 0$. Then $V_n = B(V_{n-1})$ and $V_n \subset V_{n-1}$ for all $n \geq 1$. It follows that the sequence $\{V_n\}$ must converge to a set V^U. Further, $V^U = \cap_{n=1}^{\infty} V_n$. Therefore, $V^U = B(V^U)$. We next prove that $V^U = V^*$. Indeed, by the monotonicity of operator B we get that $V^* \subset V^U$; also, $V^U \subset V^*$ since every fixed point conforms an equilibrium – given that no transversality conditions are involved in this setting. To complete the proof of the theorem, just note that $V^U \subset V^* \subset V_n$ for all $n \geq 1$. Since we have already established that $V_n \to V^U$, we get that $V_n \to V^*$. It is clear from these arguments that V^* is the largest fixed-point of operator B.

Proof of Theorems 4.1: (i) Obvious. Operator $B^{h,\Sigma}$ is monotone, $V_0 \supseteq V^*$ and $B^{h,\Sigma}(V^*) \supset V^*$.

(ii) The proof follows similar arguments as that of Theorem 3.1. Actually, $V_n^{h,\Sigma} \supset V^{*,h,\Sigma}$, and our discretized procedure allows for a finite number of set-values. Hence, pointwise convergence implies uniform convergence.

(iii) Note that operator $B^{h,\Sigma}$ converges to B as $h \to 0$ and $\Sigma \to \infty$. Since $V^* \subset V^{*,h,\Sigma}$, we get that $V^{*,h,\Sigma} \to V^*$ as $h \to 0$ and $\Sigma \to \infty$.

Proof of Theorem 4.2: The proof goes by contradiction. Since $X \times Y$ is a compact set every sequence must have a convergent subsequence. Hence, if the assertion of Theorem 4.2 is not true there is a converging sequence $\{(x_n^{h,\Sigma}, z, v_n^{h,\Sigma})\} \to (x, z, v)$ with $(x_n^{h,\Sigma}, z, v_n^{h,\Sigma}) \in graph(V_n^{h,\Sigma})$ and $d(graph(V_n^{h,\Sigma}), graph(V^*)) > \epsilon$. As $h \to 0$, $\Sigma \to \infty$, and $n \to \infty$, we

must have [cf. Theorem 3.1] that $(x, z, v) \in graph(V^*)$. However, this is in contradiction with the previous assertion that $d(graph(V_n^{h,\Sigma}), graph(V^*)) > \epsilon$ for all Σ, h, n.

Proof of Proposition 6.2: The existence of a SCE can be established by approximating the infinite-horizon economy by a sequence of finite economies. This is the strategy followed by Jones and Manuelli (1999), but their proof does not apply to sequential competitive economies. Of course, the hardest part is to provide upper bounds for equilibrium quantities over all the finite-horizon economies. These bounds follow from Proposition 6.3 below.

Hence, following Jones and Manuelli (1999), we consider the following steps for the proof of a SCE: (i) Existence of an equilibrium for a finite horizon economy. This result is covered by the general proofs of existence of competitive equilibria for economies with taxes, externalities, and incomplete markets [Arrow and Hahn (1971), Levine and Zame (1996), Mantel (1975), and Shafer and Sonneschein (1976)]. (ii) Uniform bounds for equilibrium allocations and prices of finite-horizon economies. As already pointed out, these bounds are established in Proposition 6.3 below. (iii) Existence of SEC as a limit point of finite equilibria. The preceding steps (i) and (ii) guarantee that there is a collection of vectors $\left(\{c_t^i(z^t), l_t^i(z^t), k_{t+1}^i(z^t), b^i(z^t, \xi_{t+1}^l(z^t)), K_{t+1}^i(z^t), L_t^i(z^t), r_t^i(z^t), w_t^i(z^t)\}_{i=1,2}, q(z^t, \xi_{t+1}^l(z^t))\right)_{t\geq 0}$ that can be obtained as limits of equilibria of finite economies. It is obvious that for such limiting solution the market clearing conditions must be satisfied at each z^t, and that one period-profits are maximized. Moreover, for each agent i the limiting allocation $(c_t^i(z^t), l_t^i(z^t), k_{t+1}^i(z^t), b^i(z^t, \xi_{t+1}^l(z^t))$ must satisfy the sequence of budget constraints (18), as well as the endogenous or exogenous constraints. This allocation is optimal since the discounted utility function is continuous in the product topology over the set of feasible consumption/leisure plans $\left(c_t^i\left(z^t\right), 1 - l_t^i\left(z^t\right)\right)_{t\geq 0}$ which are preferred to the endowment allocation $\left(e_t^i\left(z_t\right), 1\right)_{t\geq 0}$. This is because feasible consumption plans $\left(c_t^i\left(z^t\right)\right)_{t\geq 0}$ are bounded above, and the endowment process $\left(e_t^i\left(z_t\right)\right)_{t\geq 0}$ is bounded below by a positive quantity and the endowment of leisure is always equal to one.

Proof of Proposition 6.3: We first show that there are positive constants K^{max} and K^{min} such that for every equilibrium sequence of physical capital vectors $\left(k_{t+1}^i(z^t)\right)\right)_{t\geq 0}$ if $K^{max} \geq \sum_{i=1}^2 k_0^i(z^0) \geq K^{min}$ then $K^{max} \geq \sum_{i=1}^2 k_{t+1}^i(z^t) \geq K^{min}$ for all z^t. The existence of K^{max} follows directly from Assumption 6.1, since the marginal productivity of capital converges to zero as K goes to ∞ for every fixed $0 \leq L \leq 1$. Also, it obvious that $K^{min} \geq 0$.

We now claim that there are constants r^{max} and w^{max} such that for every equilibrium sequence of factor prices $\left(r_t^i\left(z^t\right), w_t^i\left(z^t\right)\right)_{t\geq 0}$ we have $0 \leq r_t^i(z^t) \leq r^{max}$ and $0 \leq w_t^i(z^t) \leq w^{max}$ for all z^t. The existence of w^{max} follows from continuity properties of the utility function. The household is endowed with one unit of labor. Hence, if the wage is arbitrarily high it would be optimal to consume a large amount of consumption by giving up a small quantity of leisure. If along an equilibrium path we have that r_t^i is arbitrarily large, then k_t^i must go to zero. From the Euler equation, consumption c_t^i must also go to zero. But this is not possible under either exogenous or endogenous constraints, as $e_t^i > 0$ is bounded below by a positive quantity, and in the debt constrained economy the household can switch to autarky. Moreover, using a simple arbitrage argument, we have that q_t is also bounded. Hence, the value function $J^i(k_0^i, b_0^i, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$ is well defined. As already pointed out the derivative $D_{k,b}J^i(\cdot,\cdot, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$ is continuous in $(k_0^i, b_0^i, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$.[6] By a simple notational change it follows from (18) that function J^i can be rewritten as $J^i(a_0^i, b_0^i, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$ $\mathbf{w}_0(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$, where $a_0^i = e_0^i(z_0) + (1-\tau)\, r_0 k_0^i$. Then, we can conclude that
$0 \leq D_{k,b}J^i(k_0^i, b_0^i, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0)) \leq \widehat{\gamma}$, since $e_0^i(z_0)$ is bounded below by a positive number, and all feasible vectors $(k_0^i, b_0^i, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$ lie in a compact set.

[6]Note that if b_0^i is a large negative number then the value function is well defined, but the agent will switch to autarky. In the autarky region the derivative of J^i with respect to b_0^i is zero. Hence, at the point of switching to autarky, the derivative of J^i will not be continuous but the differential is a compact correspondence.

REFERENCES

Abreu, D., D. Pierce and E. Stacchetti (1990): "Toward a theory of discounted repeated games with imperfect monitoring," *Econometrica*, 58, 1041-1063.

Arrow, K. J., and F. H. Hahn (1971): *General Competitive Analysis*, San Francisco, CA: North Holland.

Backus, D. K., P. J. Kehoe and F. E. Kydland (1992): "International real business cycles," *Journal of Political Economy*, 100, 745-775.

Bai, Y., and J. Zhang (2010): "Solving the Feldstein-Horioka puzzle with financial frictions," *Econometrica*, 78, 603-632.

Balasko, Y., and K. Shell (1980): "The overlapping generations model, I: The case of pure exchange without money," *Journal of Economic Theory*, 23, 281-306.

Beer, G. (1980): "The approximation of upper semicontinuous multifunctions by step multifunctions," *Pacific Journal of Mathematics*, 87, 11-19.

Boldrin, M., and A. Rustichini (1994): "Growth and indeterminacy in dynamic models with externalities," *Econometrica*, 62, 323-342.

Bizer, D., and K. L. Judd (1989): "Taxation under uncertainty," *American Economic Review*, 79, 331-336.

Citanna, A., and P. Siconolfi (2010): "Recursive equilibrium in stochastic overlapping-generations economies," *Econometrica*, 78, 309-347.

Coleman, W. J. (1991): "Equilibrium in a production economy with an income tax," *Econometrica*, 59, 1091-1104.

Conesa, J. C., S. Kitao and D. Krueger (1999): "Taxing capital? Not a bad idea after all," *American Economic Review*, 99, 25-48.

Datta, M., L. J. Mirman and K. L. Reffett (2002): "Existence and uniqueness of equilibrium in distorted dynamic economies with capital and labor," *Journal of Economic Theory*, 103, 377-410.

Duffie, D., J. Geanakoplos, A. Mas-Colell and A. McLennan (1994): "Stationary Markov equilibria," *Econometrica*, 62, 745-781.

Feng, Z. (2011): "Time consistent optimal fiscal policy over the business cycle," Working Paper, Department of Banking and Finance, University of Zurich.

Geanakoplos, J., M. Magill and M. Quinzii (2004): "Demography and the long-run predictability of the stock market," *Brookings Papers on Economic Activity*, Economic Studies Program, The Brookings Institution, 35, 241-326.

Gourinchas, P. O., and J. A. Parker (2002): "Consumption over the life cycle," *Econometrica*, 70, 47-89.

Grandmont, J. M. (1985): "On endogenous competitive business cycles," *Econometrica*, 53, 995–1045.

Imrohoroglu, A., S. Imrohoroglu and D. H. Joines (1995): "A life cycle analysis of social security," *Economic Theory*, 6, 83-114.

Jones, L. E., and R. E. Manuelli (1999): "Notes on the existence of equilibrium with distortions in infinite horizon economies," manuscript, University of Minnesota.

Judd, L. K., and S. Yeltekin (2010): "Computing equilibria of dynamic games," *Working Paper*, Carnegie-Mellon University.

Judd, L. K., S. Yeltekin and J. Conklin (2003): "Computing supergame equilibria," *Econometrica*, 71, 1239-1254.

Kehoe, T. J., and F. Perri (2002): "International business cycles with endogenous incomplete markets," *Econometrica*, 70, 907-928.

Kubler, F., and H. M. Polemarchakis (2004): "Stationary Markov equilibria for overlapping generations," *Economic Theory*, 24, 623-643.

Kubler, F., and K. Schmedders (2003): "Stationary equilibria in asset-pricing models with incomplete markets and collateral," *Econometrica*, 71, 1767-1795.

Kydland, F. E., and E. C. Prescott (1980): "Dynamic optimal taxation, rational expectations and optimal control," *Journal of Economic Dynamics and Control*, 2, 79-91.

Levine, D. K., and W. R. Zame (1996): "Debt constraints and equilibrium in infinite horizon economies with incomplete markets," *Journal of Mathematical Economics*, 26, 103-131.

Ljungqvist, L., and T. J. Sargent (2000): *Recursive Macroeconomic Theory*, Cambridge, MA: The MIT Press.

Mantel, R. R. (1975): "General equilibrium and optimal taxes," *Journal of Mathematical Economics*, 2, 187-200.

Marcet, A., and R. Marimon (2010): "Recursive contracts," manuscript, Universitat Pompeu Fabra, Barcelona.

Peralta-Alva, A., and M. S. Santos (2010): "Problems in the numerical simulation of models with heterogeneous agents and economic distortions," *Journal of the European Economic Association*, 8, 617-625.

Phelan, C., and E. Stacchetti (2001): "Sequential equilibria in a Ramsey tax model," *Econometrica*, 69, 1491-1518.

Rincon-Zapatero, J.-P., and M. S. Santos (2009): "Differentiability of the value function without interiority assumptions," *Journal of Economic Theory*, 144, 1948-1964.

Santos, M. S., and M. Woodford (1997): "Rational asset pricing bubbles," *Econometrica*, 65, 19-58.

Schmachtenberg, R. (1998): "Stochastic overlapping generations models with incomplete markets 1: Existence of equilibria," Discussion paper no. 363-88, Department of Economics, University of Mannheim.

Shafer, W., and H. Sonneschein (1976): "Equilibrium with externalities, commodity taxation, and lump sum transfers," *International Economic Review*, 17, 601-611.

Storesletten, K., C. I. Telmer and A. Yaron (2004): "Consumption and risk sharing over the life cycle," *Journal of Monetary Economics*, 51, 609-633.

Ventura, G. (1999): "Flat tax reform: A quantitative exploration," *Journal of Economic Dynamics and Control*, 23, 1425-1458.