A Two-Stage Latent Variable Estimation Procedure for Time-Censored Accelerated Degradation Tests

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Abstract* — Parallel constant-stress accelerated degradation testing (PCSADT) is widely used to assess the reliability of highly reliable products in a timely manner when the products' degradation can be measured. Under a time-censored PCSADT, several groups of units are tested simultaneously, but under different stress levels, until a pre-specified censoring time is reached. At this time, degradation values from the censored units, and failure times of the failed units are obtained. When the degradation follows a Wiener process where the parameters depend on the stress level through a life-stress model containing an unknown nuisance parameter, estimating this parameter often biases the maximum likelihood and least squares estimators of the lifetime parameters. In this paper, we propose a two-stage procedure to address this problem. In the first stage, we transform the data under the different stress levels of a PCSADT so that the resulting data can be considered to have been obtained under normal stress. In the second stage, we introduce a latent variable for the unobserved degradation after the failure time for each failed unit to obtain a pseudo-degradation value at the censoring time. We then use all degradation values (pseudo or observed) at the censoring time to develop latent variable estimators for all model parameters. Unlike other existing estimators, the proposed estimators are shown to be s-consistent, have closed form expressions, and are easy to interpret. We use a real example of light-emitting diodes (LEDs) to illustrate the proposed method. In addition to proving s-consistencies, we conduct a simulation study to demonstrate that the proposed estimators also perform well in finite samples.

Index Terms—**Degradation test, latent variable, decay acceleration factor, time-censored, Wiener process, inverse Gaussian distribution, s-consistent estimator.**

ACRONYMS AND ABBREVIATION

ADT	accelerated degradation test
PCSADT	parallel constant-stress ADT
IG	inverse Gaussian lifetime distribution under normal stress
EM	Expectation–Maximization algorithm
LVE	latent variable estimator
MMLE	modified maximum likelihood estimator
MEME, MEME2	modified expectation-maximization estimators
GMLE	generalized maximum likelihood estimator

NOTATION

s_l	lth stress level for $l = 0,...,k$
$W_i(t;s_l)$, $W_{li}(t)$	degradation of test unit i under s_l at time t
η, σ	drift, and diffusion parameters of standard Wiener process
α	censoring time
a	failure threshold for degradation
T_{li}	failure time of failed unit i under s_l
μ, λ	mean, and scale parameter of T_0 under s_0
μ_l, λ_l	mean, and scale parameter for T_l under s_l
$\hat{\mu}_l$, $\hat{\lambda}_l$	estimators of μ_l, and λ_l using data under s_l
$\hat{\mu}_{\text{LVE}}$, $\hat{\lambda}_{\text{LVE}}$	LVEs of μ, and λ using all data
n_l, M_l	sample size, and number of failures under s_l
N	total sample size
$\beta(s_l)$, β_l	decay-acceleration factor for s_l
$\beta(s_l;\theta)$	Arrhenius decay-acceleration factor
$\hat{\theta}_l$	estimate of θ using data under s_l
$\tilde{\alpha}_l$	time-transformed censoring time for units under s_l
$\tilde{T}_{li}$	time-transformed T_{li} for $l = 1,...k$
$f(t)$, $F(t)$	pdf, and cdf of T_0

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I. INTRODUCTION

INTENSE competition and technological advances in the manufacturing sector have resulted in products that may last a long time, making it more difficult for engineers to obtain a sufficient amount of failure data to effectively assess the products' reliabilities in a timely manner. In such cases, if the degradation of a product's critical quality characteristic can be measured, degradation measurements from an accelerated degradation test (ADT) can provide useful information about the product's reliability. This is because the failure time of a product is often defined as the first time when the product's degradation path passes a "failure" threshold. ADT has been used in various studies. For example, Meeker and Escobar [15] consider alloys, where an alloy is defined as failed when its crack length reaches 1.6 inches. Tseng et al. [32] examine light-emitting diodes (LEDs), and an LED is defined as failed when its brightness (light intensity) reaches the half-point of the initial value. For some elastomers, which are critical materials for hoses and dampers, the failure time is related to its hardness measure (Elsayed [4]).

In an ADT, the engineer needs to determine what stress levels are applied to the test units over time. Several stress loadings have been used in industry, including constant-stress, parallel constant-stress, step-stress, and continuous-stress loadings (Elsayed [4]). In the current paper, we focus on parallel constant-stress ADT (PCSADT), in which test units are divided into groups and then tested under different stress levels in parallel. Compared with other types of ADTs, a PCSADT is easier to implement and requires a less complicated statistical model for data analysis. For research on other types of stress loadings, one can refer to Tang et al. [25], Liao and Tseng [12], and Tseng et al. [29]. For an overview of degradation test models as well as the design problem, refer to Meeker and Escobar [15], Nelson [17] [18], and Yum et al. [41].

Various models have been proposed for modeling the degradation path. These include the mixed-effects non-linear regression model (Lu and Meeker [14]), the Gamma process model (Tsai et al. [27], and Tseng et al. [29]), the inverse Gaussian process model (Wang and Xu [36]), and the Wiener process model (Doksum and Hoyland [3], and Lee and Tang [8]). In this paper, we follow Doksum and Hoyland [3], and assume that the degradation path follows a Wiener process with parameters depending on the stress level via a stress-life model with an unknown (nuisance) parameter.

The Wiener process model is a growth/decay model based on Nelson's [19] cumulative exposure (CE) assumption and, unlike the Gamma and inverse Gaussian process models, it allows temporal fluctuations in the degradation process from the monotone mean degradation path. This type of fluctuation occurs in the light outputs of LEDs (see, for example, [43, Figure 3], and Vakrilov and Stoynova [33, Figure 6]) and can be caused by current fluctuations (higher current increases light intensity and lower current decreases the light intensity) and other known or unknown factors, or is simply due to measurement errors. Furthermore, unlike the standard mixed-effect model where bootstrapping and simulations are generally required in the analysis, the lifetime distribution under the Wiener process model can be represented in terms of the inverse Gaussian distribution. Therefore, the key distribution parameters such as the mean and scale parameter can be estimated from both censored and failure data using maximum likelihood methods. The estimators obtained have closed forms and are easy to interpret. Asymptotic results such as the *s*-consistency given in this paper for the parameter estimates can be obtained analytically. (An estimator is *s*-consistent if the estimator converges in probability to the parameter as the sample size approaches infinity.) The Wiener model has been used in various studies, including the resistance of self-regulating heating cables (Whitmore and Schenkelberg [38]), cracks caused by fatigue with healing (Singpurwalla [24]), light intensity of LED lamps (Tseng et al. [32]), and chip resistors (Tsai et al. [26]). Other examples can be found in Balka et al. [1], Horrocks and Thompson [5], Lehmann [9], Padgett and Tomlinson [20], Park and Padgett [21], Tseng and Peng [30], Wang [35], and Whitmore et al. [37]. The Wiener process model has been and continues to be a standard model for analyzing LED degradation data; for more recent references, see Huang et al. [6], Liu et al. [13], Tsai et al. [28], and references wherein.

When nuisance parameters exist in a model, both the Maximum Likelihood (ML) and least-squares estimations may suffer because the effect of estimating these nuisance parameters may bias the estimators of the main parameters of interest, which can lead to *s*-inconsistencies in the estimators when the number of nuisance parameters increases with the sample size (Morton [16]). In addition, the ML approach often fails to provide closed-form solutions, thus requiring the use of numerical algorithms to obtain the estimates. To overcome these difficulties, we propose a two-stage latent variable approach to obtaining latent variable estimators (LVEs) for all model parameters based on data from a time-censored PCSADT. We note that, when time censoring is used, the failure times of the censored units are unobserved, and hence are latent. Furthermore, units that failed before the censoring time are typically removed from the test; therefore, degradations occurring after "failure" times are also latent. Lee and Tang [8] use a modified expectation-maximization (EM) algorithm to predict the failure times of censored units. (An EM algorithm is an iterative method to find maximum likelihood estimators of parameters, when the model depends on unobserved latent variables.) Then, along with the observed failure times from the failed units, they propose a modified EM estimator (MEME) and a modified maximum likelihood estimator (MMLE) for both the mean and scale parameter of the lifetime distribution. Their test is an un-accelerated degradation test. The above authors also prove that the MEME for the mean failure time is *s*-consistent. The MEME of the scale parameter, however, turns out to be *s*-inconsistent; so, they propose another estimator (MEME2) to numerically reduce the asymptotic bias. In this paper, instead of predicting the unobserved failure times for the censored units, we predict the unobserved degradation values at the censoring time for the failed units. This enables us to obtain complete degradation values (pseudo or observed) at the censoring time for all units. With this data, we obtain easy-to-interpret and closed-form LVEs for the decay-acceleration factor, and the two lifetime parameters, respectively. We also prove the *s*-consistencies of these LVEs.

The rest of this paper is organized as follows. In Section II, we describe the degradation model and the PCSADT data. Section III develops the LVEs for the lifetime parameters using only data under normal stress. In Section IV, we propose the two-stage estimation procedure for obtaining LVEs of all model parameters based on all data from the PCSADT. We also prove the *s*-consistencies of the proposed LVEs. Section V provides a real example of LED lamps to illustrate the proposed two-stage estimation method. Section V also gives the results of a simulation study of the finite-sample performance of the proposed LVEs relative to several other estimators. The results show that in general the LVEs have a smaller bias and standard error than the generalized maximum likelihood estimators (GMLEs), and the estimators in Lee and Tang [8]. These results confirm Morton's [16] comments on the biasing effect when estimating a nuisance parameter. Finally, some concluding remarks are made in Section VI, and proofs of the *s*-consistencies of the proposed LVEs are given in the Appendix A.

II. DEGRADATION MODEL AND PCSADT DATA

Let $W(t;s_0)$ be the degradation of a test unit under normal stress s_0 at time *t*. The degradation path is assumed to follow a standard Wiener process (Doksum and Hoyland [3])

$$W(t;s_0)=\eta t+\sigma B(t),\quad t\geq 0\ , \tag{1}$$

where $\eta>0$, $\sigma>0$, and $B(\cdot)$ is a standard Brownian motion. The unit's lifetime T_0 is defined as the first passage time of $W(t;s_0)$ over $a\ (a>0)$:

$$T_0=\inf\{t\geq 0\,|\,W(t;s_0)\geq a\}\ . \tag{2}$$

It is well known that T_0 follows an inverse Gaussian distribution (IG) with mean, and scale parameter

$$\mu=a/\eta\ ,\quad \text{and}\quad \lambda=a^2/\sigma^2\ , \tag{3}$$

respectively. Our goal is to estimate these two lifetime parameters, based on data from a time-censored PCSADT. For further information about the IG distribution, refer to Chhikara and Folks [2], and Seshadri [23]. These references also provide the maximum likelihood estimators (MLEs) and the uniform minimum variance unbiased estimators (UMVUEs) of both μ and λ when complete data on T_0 are available.

In our PCSADT, products are first divided into *k* + 1 groups with sizes n_l, and then tested under stress s_l $(s_l\geq s_0)$, $l=0,...,k$, respectively. The total sample size is $N=\sum_{l=0}^{k}n_l$. Under s_l, the degradation path is assumed to be (Doksum and Hoyland [3], and Tsai et al. [26])

$$W(t;s_l)=\eta\cdot\beta(s_l)t+\sigma B(\beta(s_l)t)\ , \tag{4}$$

where $\eta\cdot\beta(s_l)t$ is the expected accumulated degradation at time *t*, and $\beta(\cdot)$ is the decay-acceleration factor assumed to be a non-decreasing function with $\beta(s_0)=1$ so a higher stress level is expected to result in higher accumulated degradation. This $\beta(\cdot)$ describes the life-stress relationship, and the Arrhenius model and inverse power law are quite common in practice (Meeker and Escobar [15]). Similar to (2), the failure time of $W(t;s_l)$ is defined as

$$T_l=\inf\{t\geq 0\,|\,W(t;s_l)\geq a\}\ ,\quad l=1,2,...,k\ . \tag{5}$$

Let α be the pre-specified censoring time. If test unit *i* under s_l is a failed unit, we observe its failure time, say T_{li} ($\leq\alpha$). On the other hand, if unit *i* is a censored unit, we observe its degradation value, say $W_i(\alpha;s_l)$, at α. This degradation value is sometimes written as $W_{li}(\alpha)$ for simplicity. We assume M_l (a random variable) of the n_l test units failed with failure times $T_{l1},\ldots,T_{lM_l}$, and the remaining units were censored with observed degradation values $W_{l,M_l+1}(\alpha),...,\ W_{ln_l}(\alpha)$. Then we have the following data

$$\begin{array}{llllllll}
\text{Under } s_0: & T_{01} & \ldots & T_{0M_0} & W_{0,M_0+1}(\alpha) & \ldots & W_{0n_0}(\alpha); \\
\text{Under } s_1: & T_{11} & \ldots & T_{1M_1} & W_{1,M_1+1}(\alpha) & \ldots & W_{1n_1}(\alpha); \\
\quad\ldots & & \ldots & & & \ldots & \\
\text{Under } s_k: & T_{k1} & \ldots & T_{kM_k} & W_{k,M_k+1}(\alpha) & \ldots & W_{kn_k}(\alpha).
\end{array} \tag{6}$$

This type of data has previously been considered in Padgett and Tomlinson ([20], Example 1), and Lee and Tang [8]. However, their tests are un-accelerated.

III. LATENT VARIABLE ESTIMATES USING DATA UNDER s_0

In this section, we propose a latent variable approach for estimating the lifetime parameters based only on data under s_0 (first row in (6)).

A. LVE for μ Using Data Under Normal Stress s_0

If we have a complete sample of degradation values at α from all units, these values are random samples from a normal distribution with mean $\eta\alpha$. Therefore, the sample mean is the UMVUE of $\eta\alpha$. However, units that "failed" before α are typically removed from the test in practice so that the degradations after their failure times are unobserved. We create latent variables for these unobserved degradations, as follows.

For the *i*th failed unit ($i=1,...,M_0$), the degradation value at its failure time is $W_{0i}(T_{0i})=a$ according to the definition of T_{0i} and the fact that $W_{0i}(t)$ is a continuous process. Because it is a failed unit, $\Delta W_{0i}(\alpha)\equiv W_{0i}(\alpha)-W_{0i}(T_{0i})=W_{0i}(\alpha)-a$ is a latent variable. This unobserved degradation will be estimated by its expected value in our proposed method. From (1), conditioned on the failure time $T_{0i}=t_{0i}$, the unobserved degradation can be written as

$$\Delta W_{0i}(\alpha) \equiv W_{0i}(\alpha) - W_{0i}(t_{0i}) = \eta(\alpha - t_{0i}) + \sigma(B_{0i}(\alpha) - B_{0i}(t_{0i})). \quad (7)$$

This increment follows $N(\eta(\alpha - t_{0i}), \sigma^2(\alpha - t_{0i}))$, by the independent increments property of a Brownian motion. Estimating $\Delta W_{0i}(\alpha)$ by its expectation above, we obtain a pseudo-degradation value $W_{0i}(\alpha) = W_{0i}(T_{0i}) + \Delta W_{0i}(\alpha) = a + \eta(\alpha - T_{0i})$ for $i = 1, ..., M_0$. Then, along with the observed degradation values at α from the censored units, we have a complete pseudo sample of degradation values $(W_{01}(\alpha), \ldots, W_{0M_0}(\alpha), W_{0,M_0+1}(\alpha), \ldots, W_{0n_0}(\alpha))$ for all units. Motivated by the UMVUE discussed earlier, we obtain the LVE of η (denoted by $\hat{\eta}_0$) by solving $\hat{\eta}$:

$$\hat{\eta}\alpha = \frac{\sum_{i=1}^{n_0} W_{0i}(\alpha)}{n_0} = \frac{\sum_{i=1}^{M_0}\left(a + \hat{\eta}\cdot(\alpha - T_{0i})\right)}{n_0} + \frac{\sum_{i=M_0+1}^{n_0} W_{0i}(\alpha)}{n_0}.$$

That is,

$$\hat{\eta}_0 = \frac{\sum_{i=M_0+1}^{n_0} W_{0i}(\alpha) + aM_0}{(n_0 - M_0)\alpha + \sum_{i=1}^{M_0} T_{0i}}. \quad (8)$$

In view of (3), the LVE of μ is given by

$$\hat{\mu}_0 = \frac{\sum_{i=1}^{M_0} T_{0i} + (n_0 - M_0)\alpha}{M_0 + \sum_{i=M_0+1}^{n_0} W_{0i}(\alpha)/a}. \quad (9)$$

The LVE in (9) turns out to be identical to the MEME and MMLE proposed by Lee and Tang [8] but under a different approach, and is an *s*-consistent estimator of μ. Furthermore, $\hat{\mu}_0$ is a weighted average of two average "lifetimes": the average observed lifetime of the failed units ($\sum_{i=1}^{M_0} T_{0i} / M_0$), and the average "predicted (pseudo-) lifetime" of the censored units $(n_0 - M_0)\alpha / (\sum_{i=M_0+1}^{n_0} W_{0i}(\alpha)/a)$. To explain the latter, we note that the average predicted lifetime of all censored units is $\sum_{i=M_0+1}^{n_0}\left(\alpha + (a - W_{0i}(\alpha))/\eta\right)/(n_0 - M_0)$, where $(a - W_{0i}(\alpha))/\eta$ is the predicted (pseudo-) remaining time after α for unit *i*. Estimating the unknown slope η by $(\sum_{i=M_0+1}^{n_0} W_{0i}(\alpha)/(n_0 - M_0) - 0)/(\alpha - 0)$, we obtain the average predicted (pseudo-) lifetime of the censored units mentioned earlier. The two weights used to obtain $\hat{\mu}_0$ in (9) are $M_0/(M_0 + \sum_{i=M_0}^{n_0} W_{i0}(\alpha)/a)$, and $(\sum_{i=M_0+1}^{n_0} W_{i0}(\alpha)/a)/(M_0 + \sum_{i=M_0+1}^{n_0} W_{i0}(\alpha)/a))$, respectively. M_0 is the number of failed units. For the censored units, $W_{i0}(\alpha) < a$ so that $W_{i0}(\alpha)/a$ can be interpreted as (pseudo-) censoring ratio and $\sum_{i=M_0+1}^{n_0} W_{i0}(\alpha)/a$ as the (pseudo-) total number of censored units. So, the weights are the proportion of failed units and the (pseudo-) proportion of censored units, respectively, in the experiment.

B. LVE for λ Using Data Under Normal Stress s_0

To estimate λ, we first use a modified EM algorithm with M- and E-steps to obtain an LVE of σ^2, as follows.

Step 1. Let $\hat{\sigma}^2_{(0)}$ be an initial estimate of σ^2.

Step 2 (M-Step). If we have complete random degradation values $W_{0i}(\alpha)$, these values form a random sample from a normal distribution with mean $\eta\alpha$, and variance $\alpha\sigma^2$. Hence, assuming η is known, the MLE of σ^2 is given by

$$\begin{aligned}\hat{\sigma}^2 &= \frac{1}{n_0\alpha}\sum_{i=1}^{n_0}(W_{0i}(\alpha) - \eta\alpha)^2\\ &= \frac{1}{n_0\alpha}\left(\sum_{i=1}^{M_0}(W_{0i}(\alpha) - \eta\alpha)^2 + \sum_{i=M_0+1}^{n_0}(W_{0i}(\alpha) - \eta\alpha)^2\right)\\ &= \frac{1}{n_0\alpha}\left(\sum_{i=1}^{M_0}(a + \Delta W_{0i}(\alpha) - \eta\alpha)^2 + \sum_{i=M_0+1}^{n_0}(W_{0i}(\alpha) - \eta\alpha)^2\right)\\ &= \frac{1}{n_0\alpha}\left(\sum_{i=1}^{M_0}\left((a - \eta T_{0i})^2 + 2(a - \eta T_{0i})\sigma\Delta B_{0i}(\alpha) + \sigma^2\Delta B_{0i}(\alpha)^2\right)\right.\\ &\qquad \left. + \sum_{i=M_0+1}^{n_0}(W_{0i}(\alpha) - \eta\alpha)^2\right), \quad (10)\end{aligned}$$

where $\Delta B_{0i}(\alpha) \equiv B_{0i}(\alpha) - B_{0i}(T_{0i})$ are independent of the failure times T_{0i}. The last equality in (10) follows from (7).

Step 3 (E-Step). Because $\Delta B_{0i}(\alpha)$ and $\Delta B_{0i}(\alpha)^2$ in (10) are unobserved (because $\Delta W_{0i}(\alpha)$ are unobserved), they are estimated by their respective expected values: $E_{B_{0i}(\cdot)}(\Delta B_{0i}(\alpha)) = 0$, and $E_{B_{0i}(\cdot)}(\Delta B_{0i}(\alpha)^2) = \alpha - T_{0i}$. In this step, we also replace σ^2 on the right side of (10) with its initial estimate ($\hat{\sigma}^2_{(0)}$) from Step 1 to obtain the first updated estimate of σ^2:

$$\hat{\sigma}^2_{(1)} = \hat{\sigma}^2_{(0)}\left(\frac{1}{n_0\alpha}\sum_{i=1}^{M_0}(\alpha - T_{0i})\right) + \frac{1}{n_0\alpha}\left(\sum_{i=1}^{M_0}(a - \eta T_{0i})^2 + \sum_{i=M_0+1}^{n_0}(W_{0i}(\alpha) - \eta\alpha)^2\right) \quad (11)$$

Step 4. $\hat{\sigma}^2_{(1)}$ in (11) will replace $\hat{\sigma}^2_{(0)}$ in Step 2 (M-Step) and Step 3 (E-Step) to obtain another updated estimate $\hat{\sigma}^2_{(2)}$. This process is repeated so we have

$$\hat{\sigma}^2_{(k+1)} = \hat{\sigma}^2_{(k)}\left(\frac{1}{n_0\alpha}\sum_{i=1}^{M_0}(\alpha - T_{0i})\right) + \frac{1}{n_0\alpha}\left(\sum_{i=1}^{M_0}(a - \eta T_{0i})^2 + \sum_{i=M_0+1}^{n_0}(W_{0i}(\alpha) - \eta\alpha)^2\right), \; k = 0, 1, \ldots.$$

Assume $\hat{\sigma}^2_{(k)} \to \hat{\sigma}^2$. Then, by taking the limits on both sides and solving for $\hat{\sigma}^2$, we obtain the proposed LVE of σ^2. Because this LVE uses only data under s_0, it will be denoted by $\hat{\sigma}_0^2$:

$$\hat{\sigma}_0^2 = \frac{\sum_{i=1}^{M_0}(a-\eta T_{0i})^2 + \sum_{i=M_0+1}^{n_0}(W_{0i}(\alpha)-\eta\alpha)^2}{\sum_{i=1}^{M_0} T_{0i} + \alpha(n_0 - M_0)}. \quad (12)$$

This $\hat{\sigma}_0^2$ is also easy to interpret because it is a weighted average of two estimates of σ^2. First we note that the second moment of the degradation values $\sum_{i=M_0+1}^{n_0}(W_{0i}(\alpha)-\eta\alpha)^2 / (n_0 - M_0)$ from the censored units is an MLE of the population variance $\alpha\sigma^2$, assuming η and M_0 are given. Second, the sample mean lifetime $\sum_{i=1}^{M_0} T_{0i}/M_0$, and the sample variance $\sum_{i=1}^{M_0}(T_{0i}-\mu)^2/M_0$, from the failed units are estimates of the *IG* mean μ (= a/η), and variance μ^3/λ (= $\sigma^2(\mu^3/a^2)$), respectively. With some algebraic manipulation, we obtain a second estimate of σ^2 as $\sum_{i=1}^{M_0}(a-\eta T_{0i})^2 \big/ \sum_{i=1}^{M_0} T_{0i}$. The weights used to obtain $\hat{\sigma}_0^2$ above are $\alpha(n_0 - M_0)/(\sum_{i=1}^{M_0} T_{0i} + \alpha(n_0 - M_0))$ and $\sum_{i=1}^{M_0} T_{0i} / (\sum_{i=1}^{M_0} T_{0i} + \alpha(n_0 - M_0))$. Note that $\sum_{i=1}^{M_0} T_{0i}$ is the total testing time (or observed lifetime) of the failed units, while $\alpha(n_0 - M_0)$ is the total testing time (or (pseudo-) lifetime) of the censored units. So the two weights are the fractions of time used in testing failed and censored units, respectively.

Finally, estimating the unknown η in (12) by $\hat{\eta}_0$, we obtain the LVE of λ (use (3)):

$$\hat{\lambda}_0 = \frac{\sum_{i=1}^{M_0} T_{0i} + \alpha(n_0 - M_0)}{\sum_{i=1}^{M_0}(1 - T_{0i}/\hat{\mu}_{\text{LVE}})^2 + \sum_{i=M_0+1}^{n_0}(W_{0i}(\alpha)/a - \alpha/\hat{\mu}_{\text{LVE}})^2}. \quad (13)$$

While the MEME of λ in Lee and Tang [8] is not *s*-inconsistent, the LVEs in (9) and (13) provide easy-to-interpret, *s*-consistent, and closed-form estimators for both IG lifetime parameters. Proof of the *s*-consistency of $\hat{\lambda}_0$ for λ is given in the Appendix A.

IV. TWO-STAGE LATENT VARIABLE ESTIMATION USING ALL DATA

In this section, we propose a two-stage estimation procedure using all PCSADT data in (6). The approach is to first transform the data in (6) so that the resulting data can be viewed as data all obtained under s_0. Then, we follow the development in Section III to obtain our proposed LVEs for all model parameters in the second stage.

A. *First Stage: Estimation of Decay-Acceleration Factors $\beta(s_l)$*

Under s_l, the accelerated degradation process in (4) can be considered as a standard Wiener process (1) but with a drift parameter $\eta_l \equiv \eta \cdot \beta(s_l)$. Hence from (8), we obtain an LVE for η_l as

$$\hat{\eta}_l = \frac{\sum_{i=M_l+1}^{n_l} W_{li}(\alpha) + aM_l}{(n_l - M_l)\alpha + \sum_{i=1}^{M_l} T_{li}}, \quad l = 0,1,\ldots,k. \quad (14)$$

In view of (3), an LVE of the mean failure time under s_l (say $\mu_l = a/\eta_l$) is given by $\hat{\mu}_l = a/\hat{\eta}_l$. Furthermore, from the assumption that $\beta(s_0)=1$, we obtain $\eta = \eta_0$. Consequently, an estimator of $\beta(s_l)$ is given by

$$\hat{\beta}(s_l) = \frac{\hat{\eta}_l}{\hat{\eta}_0} = \frac{aM_l + \sum_{i=M_l+1}^{n_l} W_{li}(\alpha)}{(n_l - M_l)\alpha + \sum_{i=1}^{M_l} T_{li}} \Bigg/ \frac{aM_0 + \sum_{i=M_0+1}^{n_0} W_{0i}(\alpha)}{(n_0 - M_0)\alpha + \sum_{i=1}^{M_0} T_{0i}}, \quad l = 1,\ldots,k. \quad (15)$$

These estimates require no explicit functional form for $\beta(\cdot)$, which can be difficult to obtain because it often depends on the material features of the product, and the type of stress used. Now, based on our discussion of the *s*-consistency of the LVE of mean lifetime in the last section, $\hat{\eta}_l$ is *s*-consistent for η_l, and consequently $\hat{\beta}(s_l)$ is an *s*-consistent estimator of $\beta(s_l)$ for $l = 1,\ldots,k$, by the Slusky's theorem (Lehmann [10], p.70).

For certain products, on the other hand, the decay-acceleration functions have long been established. For example, Padgett and Tomlison [20] consider both the power law and the Arrhenius model for $\beta(\cdot)$ in their experiment for certain carbon-film resistors, where temperature is used as the stress. Indeed, the Arrhenius model is commonly used in accelerated life tests and ADTs, when the stress is temperature (Meeker and Escobar [15]). A one-parameter Arrhenius model has the following functional form

$$\beta(s;\theta) = \exp\left(\frac{\theta}{k_b}\left(\frac{1}{273.15+s_0} - \frac{1}{273.15+s}\right)\right), \quad s \ge s_0, \quad (16)$$

where θ is unknown, the temperature s is in Celsius scale (^{o}C), and k_b = 1/11,605 is the Boltzmann's constant. Then, θ can be expressed as

$$\theta = \beta^{-1}(\beta; s) = \frac{k_b \ln(\beta)}{\left(1/(273.15+s_0) - 1/(273.15+s)\right)}, \quad (17)$$

where $\beta^{-1}(\cdot; s)$ is the inverse function of (16), and β denotes a value of $\beta(s;\theta)$.

Now, because $\hat{\beta}(s_l)$ in (15) is *s*-consistent for $\beta(s_l;\theta)$, and $\beta^{-1}(\cdot;s)$ is a continuous function, each $\hat{\theta}_l \equiv \beta^{-1}(\hat{\beta}(s_l);s_l)$ is an *s*-consistent estimator of $\beta^{-1}(\beta(s_l;\theta),s_l)=\theta$ for $l=1,\ldots,k$.

To estimate θ using all data, we propose the asymptotically best linear unbiased estimator as follows

$$\hat{\theta}_{\text{LVE}} = \sum_{l=1}^{k} a_l \hat{\theta}_l , \quad (18)$$

where a_l are obtained by minimizing the asymptotic variance of (18), subject to $\sum_{l=1}^{k} a_l = 1$ so that $\hat{\theta}_{\text{LVE}}$ is also *s*-consistent for θ. To compute a_l, we write

$$\hat{\theta}_l = \beta^{-1}(\hat{\beta}(s_l);s_l) = \frac{k_b \ln(\hat{\beta}(s_l))}{1/(c+s_0) - 1/(c+s_l)}$$
$$= h_l\left(\ln(\hat{\mu}_0) - \ln(\hat{\mu}_l)\right), \text{ say}, \quad (19)$$

where c = 273.15, and $h_l \equiv k_b/\left(1/(c+s_0) - 1/(c+s_l)\right)$. The estimator (18) becomes

$$\sum_{l=1}^{k} a_l \hat{\theta}_l = \sum_{l=1}^{k} a_l h_l \left(\ln(\hat{\mu}_0) - \ln(\hat{\mu}_l)\right) = \sum_{l=0}^{k} b_l \ln(\hat{\mu}_l), \quad (20)$$

where $\ln(\hat{\mu}_l)$ are *s*-independent, $b_l = -h_l a_l$ for $l=1,\ldots,k$, and $b_0 = \sum_{l=1}^{k} h_l a_l = -\sum_{l=1}^{k} b_l$. Denote $\delta_l^2 \equiv Var(\ln(\hat{\mu}_l))$. Then from (20), the variance of (18) is

$$V \equiv Var\left(\sum_{l=1}^{k} a_l \hat{\theta}_l\right) = Var\left(\sum_{l=0}^{k} b_l \ln(\hat{\mu}_l)\right)$$
$$= \sum_{l=0}^{k} b_l^2 Var\left(\ln(\hat{\mu}_l)\right) = \sum_{l=1}^{k} b_l^2 \delta_l^2 + \left(\sum_{l=1}^{k} b_l\right)^2 \delta_0^2. \quad (21)$$

Because $\sum_{l=1}^{k} b_l/h_l = -1$, we have $b_k = -h_k\left(1 + \sum_{l=1}^{k-1} b_l/h_l\right)$, and consequently

$$V = \left(\sum_{l=1}^{k-1} b_l^2 \delta_l^2 + h_k^2\left(1 + \sum_{l=1}^{k-1}\frac{b_l}{h_l}\right)^2 \delta_k^2\right) + \left(\sum_{l=1}^{k-1} b_l - h_k\left(1 + \sum_{l=1}^{k-1}\frac{b_l}{h_l}\right)\right)^2 \delta_0^2 . \quad (22)$$

In general, an iterative procedure is required to numerically obtain b_l and a_l when $k \geq 3$. However, for a three-level PSCADT with k = 2 (the example we consider in Section V), by taking the derivatives with respect to b_l to (22), we can obtain closed-form expressions for b_l and hence a_l:

$$a_1 = -\frac{b_1}{h_1} = \frac{\frac{h_2^2}{h_1^2}\frac{\delta_2^2}{\delta_1^2} - \left(1 - \frac{h_2}{h_1}\right)\frac{\delta_0^2}{\delta_1^2}\frac{h_2}{h_1}}{1 + \frac{h_2^2}{h_1^2}\frac{\delta_2^2}{\delta_1^2} + \left(1 - \frac{h_2}{h_1}\right)^2 \frac{\delta_0^2}{\delta_1^2}}$$

, and $a_2 = 1 - a_1$, where a_1 can be negative if δ_2^2 is too small, or δ_0^2 is too large. We provide some remarks in Appendix B to give an alternative method for obtaining the LVE of θ, or $\underline{\theta}$ if one wants to use our method for cases with multiple stress factors.

Notice that both b_l and a_l depend on the unknown δ_l^2. Using the Delta method (Meeker and Escobar [15], pp. 619-620), $\delta_l^2 \equiv Var(\ln(\hat{\mu}_l)) \cong Var\left(\hat{\mu}_l / E(\hat{\mu}_l)\right) = Var(\hat{\mu}_l)/E^2(\hat{\mu}_l)$, $l = 0, 1, \ldots, k$. Above, both the numerator and denominator also need to be estimated. Because $\hat{\mu}_l$ is *s*-consistent for μ_l, $E(\hat{\mu}_l)$ will be estimated by $\hat{\mu}_l$. Furthermore, from Lee and Tang [8], we can obtain

$$Var(\hat{\mu}_l) \approx \left\{\frac{n_l \lambda_l}{\mu_l^4}\left[\mu_l\left(\Phi(A_l) - \exp\left(\frac{2\lambda_l}{\mu_l}\right)\Phi(-B_l)\right) + \alpha\left(1 - F_l(\alpha)\right)\right]\right\}^{-1},$$

where $A_l = \sqrt{\lambda_l/\alpha}(\alpha/\mu_l - 1)$, $B_l = \sqrt{\lambda_l/\alpha}(\alpha/\mu_l + 1)$, and $F_l(\alpha)$ is the individual failure probability under stress s_l (Chhikara and Folks [2]):

$$F_l(\alpha) = \Phi\left(\sqrt{\frac{\lambda_l}{\alpha}}\left(\frac{\alpha}{\mu_l} - 1\right)\right) + \exp\left(\frac{2\lambda_l}{\mu_l}\right)\Phi\left(-\sqrt{\frac{\lambda_l}{\alpha}}\left(\frac{\alpha}{\mu_l} + 1\right)\right). \quad (23)$$

Again, the unknown μ_l in (23) is estimated by $\hat{\mu}_l$. As we will show in the next section, λ_l can be estimated by

$$\hat{\lambda}_l = \frac{\sum_{i=1}^{M_l} T_{li} + \alpha(n_l - M_l)}{\sum_{i=1}^{M_l}(1 - T_{li}/\hat{\mu}_l)^2 + \sum_{i=M_l+1}^{n_l}(W_{li}(\alpha)/a - \alpha/\hat{\mu}_l)^2}. \quad (24)$$

With the above results, we can now estimate a_l to obtain $\hat{\theta}_{\text{LVE}}$ in (18).

B. *Second Stage: Estimation of* (μ,λ) *Using All Data in (6)*

In this section, we first transform the data in (6) so that the resulting data can be viewed as data all obtained under s_0. Again, we note that the value of $W(t;s_l)$ in (4) at time *t* can be viewed as a value of $W(\beta_l t;s_0)$ but at time $\beta_l t$ (β_l stands for $\beta(s_l)$ for simplicity). Hence, if we transform the time from *t* to $\tilde{t}_l \equiv \beta_l t$, then the transformed failure times $\tilde{T}_l \equiv \beta_l T_l$ are *s*-independent random variables such that

$$\tilde{T}_l \sim IG(\mu,\lambda),\ l = 0,\ldots,k. \quad (25)$$

The corresponding censoring times are

$$\tilde{\alpha}_l \equiv \beta_l \cdot \alpha,\ \ l = 0,\ldots,k. \quad (26)$$

In terms of the transformed time, the degradation value $W_{li}(\alpha)$ in (6) is rewritten as $W_{li}(\tilde{\alpha}_l)$, even though they have the same value. Estimating β_l in (26) by $\hat{\beta}_l$ in (15), we have the following data, all under s_0:

$$\begin{array}{llllll} T_{01} & \dots & T_{0M_0} & W_{0,M_0+1}(\alpha) & \dots & W_{0n_0}(\alpha), \\ \tilde{T}_{11} & \dots & \tilde{T}_{1M_1} & W_{1,M_1+1}(\tilde{\alpha}_1) & \dots & W_{1n_1}(\tilde{\alpha}_1), \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \tilde{T}_{k1} & \dots & \tilde{T}_{kM_k} & W_{k,M_k+1}(\tilde{\alpha}_k) & \dots & W_{kn_k}(\tilde{\alpha}_k). \end{array} \tag{27}$$

The method given in Section III can then be extended to obtain the LVEs of μ, and λ, based on all data in (27).

B.1. LVE for μ Based on All Data (27)

To estimate μ, we first estimate the drift parameter η, as before. If we have all degradation data, then

$$\frac{\sum_{i=1}^{M_l} W_{li}(\tilde{\alpha}_l) + \sum_{i=M_l+1}^{n_l} W_{li}(\tilde{\alpha}_l)}{n_l\tilde{\alpha}_l} \sim N\left(\eta, \frac{\sigma^2}{n_l\tilde{\alpha}_l}\right), \ l = 0,1,\dots,k.$$

Hence, the optimal linear unbiased estimator of η under quadratic loss is given by (Kagan et al. [7], p. 227)

$$\hat{\eta} = \frac{\sum_{l=0}^{k}\left(\left(\frac{\sum_{i=1}^{M_l} W_{li}(\tilde{\alpha}_l) + \sum_{i=M_l+1}^{n_l} W_{li}(\tilde{\alpha}_l)}{n_l\tilde{\alpha}_l}\right)\Big/\frac{1}{n_l\tilde{\alpha}_l}\right)}{\sum_{l=0}^{k} 1\Big/\left(\frac{1}{n_l\tilde{\alpha}_l}\right)} = \frac{\sum_{l=0}^{k}\left(\sum_{i=1}^{M_l} W_{li}(\tilde{\alpha}_l) + \sum_{i=M_l+1}^{n_l} W_{li}(\tilde{\alpha}_l)\right)}{\sum_{l=0}^{k} n_l\tilde{\alpha}_l}.$$

Because $W_{li}(\tilde{\alpha}_l) = W_{li}(\tilde{T}_{li}) + (W_{li}(\tilde{\alpha}_l) - W_{li}(\tilde{T}_{li})) = a + \Delta W_{li}(\tilde{\alpha}_l)$ where $\Delta W_{li}(\tilde{\alpha}_l)$ is latent, we estimate this latent variable by its expectation $\eta(\tilde{\alpha}_l - \tilde{T}_{li})$ (with respect to the distribution of $W_{li}(\cdot)$) to obtain

$$\hat{\eta}\left(1 - \frac{\sum_{l=0}^{k}\sum_{i=1}^{M_l}(\tilde{\alpha}_l - \tilde{T}_{li})}{\sum_{l=0}^{k} n_l\tilde{\alpha}_l}\right) = \frac{\sum_{l=0}^{k}\left(\sum_{i=M_l+1}^{n_l} W_{li}(\tilde{\alpha}_l) + aM_l\right)}{\sum_{l=0}^{k} n_l\tilde{\alpha}_l}.$$

Then, from the fact that $W_{li}(\tilde{\alpha}_l)$ and $W_{li}(\alpha)$ have the same value, we have

$$\hat{\eta} = \frac{\sum_{l=0}^{k}\left(\sum_{i=M_l+1}^{n_l} W_{li}(\alpha) + aM_l\right)}{\sum_{l=0}^{k} n_l\tilde{\alpha}_l - \sum_{l=0}^{k}\left(M_l\tilde{\alpha}_l - \sum_{i=1}^{M_l}\tilde{T}_{lj}\right)} = \frac{\sum_{l=0}^{k}\left(\sum_{i=M_l+1}^{n_l} W_{li}(\alpha) + aM_l\right)}{\sum_{l=0}^{k}(n_l - M_l)\tilde{\alpha}_l + \sum_{l=0}^{k}\sum_{i=1}^{M_l}\tilde{T}_{lj}}.$$

Finally, estimating $\tilde{\alpha}_l$, and $\tilde{T}_{li}$ by $\hat{\beta}_l\alpha$, and $\hat{\beta}_l T_{li}$, respectively, we obtain the LVE of μ as

$$\hat{\mu}_{\text{LVE}} = \frac{\sum_{l=0}^{k}\hat{\beta}_l\left((n_l - M_l)\alpha + \sum_{i=1}^{M_l} T_{li}\right)}{\sum_{l=0}^{k}\left(\sum_{i=M_l+1}^{n_l} W_{li}(\alpha)/a + M_l\right)}. \tag{28}$$

This LVE is again a linear combination of LVEs of μ from different stress levels, adjusted by $\hat{\beta}_l$. Since $\hat{\mu}_{\text{LVE}}$ is the LVE of the *normal* mean lifetime (under s_0), we need $\hat{\beta}_l$ to adjust the *accelerated* lifetime T_{li} so that $\hat{\beta}_l T_{li}$ $(=\tilde{T}_{li})$ is the predicted normal lifetime for unit *i*. Furthermore, using the same discussion for $\hat{\mu}_0$ in (9), we see that the weights for the linear combination are each level's relative (pseudo-) number of test units. $\hat{\mu}_{\text{LVE}}$ is also the total (pseudo-) predicted normal lifetime divided by the total (pseudo-) number of test units. Finally the proposed $\hat{\mu}_{\text{LVE}}$ in (28) is an *s*-consistent estimator of μ (proof is given in the Appendix A).

B.2. LVE for λ Based on All Data (27)

To estimate λ, we first estimate σ^2. We normalize $W_{li}(\tilde{\alpha}_l)$ such that $W_{li}(\tilde{\alpha}_l)/\sqrt{\tilde{\alpha}_l} \sim N(\sqrt{\tilde{\alpha}_l}\eta, \sigma^2)$ for $i = M_l + 1, \dots, n_l$. Temporally assuming η and β_l are known, we follow the same argument as in Section III to obtain the estimate of σ^2 as

$$\hat{\sigma}^2 = \frac{\sum_{l=0}^{k}\left[\sum_{i=1}^{M_l}\left(a - \eta\tilde{T}_{li}\right)^2 + \sum_{i=M_l+1}^{n_l}\left(W_{li}(\tilde{\alpha}_l) - \tilde{\alpha}_l\eta\right)^2\right]\Big/\tilde{\alpha}_l}{\sum_{l=0}^{k}\left[\sum_{i=1}^{M_l}\tilde{T}_{li} + (n_l - M_l)\tilde{\alpha}_l\right]\Big/\tilde{\alpha}_l}. \tag{29}$$

This estimator is a weighted average of separate independent estimators of σ^2 and can be rewritten as

$$\hat{\sigma}^2 = \sum_{l=0}^{k}\left(\frac{\sum_{i=1}^{M_l} T_{li} + (n_l - M_l)\alpha_l}{\sum_{l=0}^{k}\left(\sum_{i=1}^{M_l} T_{li} + (n_l - M_l)\alpha_l\right)}\right)\hat{\sigma}_l^2 = \sum_{l=0}^{k} w_l\hat{\sigma}_l^2, \text{ say,}$$

where $\hat{\sigma}_l^2$ is the LVE of σ^2 using only data under stress level s_l (cf. (12)):

$$\hat{\sigma}_l^2 \equiv \frac{\sum_{i=1}^{M_l}\left(a - \eta\tilde{T}_{li}\right)^2 + \sum_{i=M_l+1}^{n_l}\left(W_{li}(\tilde{\alpha}_l) - \tilde{\alpha}_l\eta\right)^2}{\sum_{i=1}^{M_l}\tilde{T}_{li} + (n_l - M_l)\tilde{\alpha}_l}.$$

Hence, using the same discussion for $\hat{\sigma}_0^2$ in (12), we see that the weights w_l are proportions of the total time used in testing the units under stress levels s_l, $l = 0,1,\dots,k$. Note that $\hat{\lambda}_l$ in (24) is $\hat{\lambda}_l = a^2/\hat{\sigma}_l^2$ with η estimated by $\hat{\eta}_l$.

Using (3), the conditional estimate of λ is

$$\hat{\lambda}=\frac{\sum_{l=0}^{k}\left[\sum_{i=1}^{M_l}\tilde{T}_{li}+(n_l-M_l)\tilde{\alpha}_l\right]\Big/\tilde{\alpha}_l}{\sum_{l=0}^{k}\left[\sum_{i=1}^{M_l}\left(1-\tilde{T}_{li}/\mu\right)^2+\sum_{i=M_l+1}^{n_l}\left(W_{li}(\tilde{\alpha}_l)/a-\tilde{\alpha}_l/\mu\right)^2\right]\Big/\tilde{\alpha}_l}.$$

Finally, we estimate the unknown μ, and β_l by their s-consistent estimators $\hat{\mu}_{\mathrm{LVE}}$, and $\hat{\beta}_l$, respectively. The proposed LVE of λ using all data is given by (recall $W_{li}(\tilde{\alpha}_l)$ and $W_{li}(\alpha)$ have the same value)

$$\hat{\lambda}_{\mathrm{LVE}}=\frac{\sum_{l=0}^{k}\left[\sum_{i=1}^{M_l}T_{li}+(n_l-M_l)\alpha\right]}{\sum_{l=0}^{k}\frac{1}{\hat{\beta}_l}\left[\sum_{i=1}^{M_l}\left(1-\hat{\beta}_l T_{li}/\hat{\mu}_{\mathrm{LVE}}\right)^2+\sum_{i=M_l+1}^{n_l}\left(W_{li}(\alpha)/a-\hat{\beta}_l\alpha/\hat{\mu}_{\mathrm{LVE}}\right)^2\right]}. \tag{30}$$

To interpret $\hat{\lambda}_{\mathrm{LVE}}$, we note that the numerator is the sum of numerators in (13) over all stress levels, and that the denominator is a weighted sum of denominators in (13) also over all stress levels, where the weights are $1/\hat{\beta}_l$. This $\hat{\lambda}_{\mathrm{LVE}}$ is an s-consistent estimator of λ (proof of this result is also given in the Appendix A).

To obtain the LVEs of the mean and scale parameter of the *accelerated* failure time distribution under a specific stress level s_l, we note that under s_l, the failure time $T_l=T/\beta(s_l)$ $\sim IG(\mu_l,\lambda_l)$ with $\mu_l=\mu/\beta(s_l)$ and $\lambda_l=\lambda/\beta(s_l)$ (see Chhikara and Folks [2]). Hence, the LVEs of μ_l and λ_l based on all data are $\hat{\mu}_{l,\mathrm{LVE}}=\hat{\mu}_{\mathrm{LVE}}/\hat{\beta}(s_l)$ and $\hat{\lambda}_{l,\mathrm{LVE}}=\hat{\lambda}_{\mathrm{LVE}}/\hat{\beta}(s_l)$, respectively. If one does not want to use data from a particular stress level, simply let the sample size be 0 for that level in our procedure.

V. LED EXAMPLE

In this section, we use a real LED dataset to illustrate the proposed method. The lifetime of an LED is related to its normalized light intensity or brightness (denoted $L_{li}(t)$). Specifically, the lifetime of an LED lamp is defined as the first time when its $L_{li}(t)$ reaches a certain "failure" threshold, which is typically 50% of its starting value of $L_{li}(0)=1$.

The procedure for conducting an LED accelerated degradation test is fairly standard (see, for example, Huang et al. [6], Wang et al. [34], and [44]). The main test equipment is a high temperature aging degradation chamber. As to stress factors, IES LM-80-08 [42], which is an industry standard developed by the Illuminating Engineering Society of North America and sponsored by the U.S. Department of Energy, suggests using only one stress factor, namely temperature, at three different levels when analyzing the lumen degradation and the lifetime of LEDs. Our data are obtained from one of the leading LED manufacturers in Taiwan (see Tseng et al. [31]),

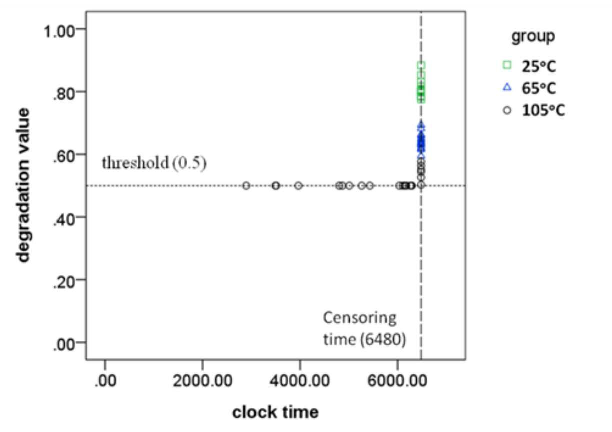

Fig. 1. The Original LED Data from a Three-level PCSADT.

based on a three-level (k = 2) PCSADT with temperatures (in °C) set at (s_0,s_1,s_2) = (25, 65, 105). The electric current was set at the normal level of 10 mA. All other factors were also set at their respective normal levels. The censoring time for this PCSADT is 6,480 hours. The sample sizes for the three stress levels are n_0 = 15, n_1 = 18, and n_2 = 24, respectively. Under s_0 = 25°C, and s_1 = 65°C, no LED lamp failed; so $m_0=m_1=0$. However, under s_2 = 105°C, m_2 = 18 units failed. Failure times and degradation values at the censoring time are depicted in Fig. 1.

According to Yu and Tseng [39] [40], Tseng et al. [32], and Liao and Elsayed [11], $L_{li}(t)$ can be linearized so that $W_{li}(t)\equiv-\ln(L_{li}(t^{0.6}))$ follows a Wiener process (4). Consequently, failure times, degradation values, and the censoring time shown in Fig. 1 are transformed. These transformed values are our T_{li} and $W_{li}(t)$, with $\alpha=6480^{0.6}$ = 193.62. The transformed failure threshold for $W_{li}(t)$ is a = $-\ln(0.5)=0.6932$. Because temperature is the stress factor, the parameter-stress relationship is modeled by an Arrhenius law in (16) with an unknown θ.

A. *Various Estimates of the Model Parameters*

Following the proposed two-stage LV procedure, we obtain $\hat{\theta}_{\mathrm{LVE}}$ = 0.1499 from (18) in the first stage. Then, the estimated decay-acceleration factor is $\hat{\beta}_l=\beta(s_l;\hat{\theta}_{\mathrm{LVE}})=\exp\{(0.1499/k_b)$ $\times(1/(273.15+25)-1/(273.15+s_l))\}$ with $\hat{\beta}_2$ =1.9941 and $\hat{\beta}_3$ =3.4361. In the second stage, we first multiply these estimated decay-acceleration factors to the respective failure times (under different stress levels) to obtain $\tilde{T}_{li}$ in (27). The transformed censoring times are $\tilde{\alpha}_1$ = 386.093, and $\tilde{\alpha}_2$ = 665.305. These transformed data are depicted in Fig. 2. Using these data, we obtain $\hat{\mu}_{\mathrm{LVE}}$ = 618.97, and $\hat{\lambda}_{\mathrm{LVE}}$ = 42,676.2 (Table I).

We now compare our two-stage LVEs with several existing estimators. The first set is the General Maximum Likelihood Estimators (GMLEs) below. The likelihood function for the

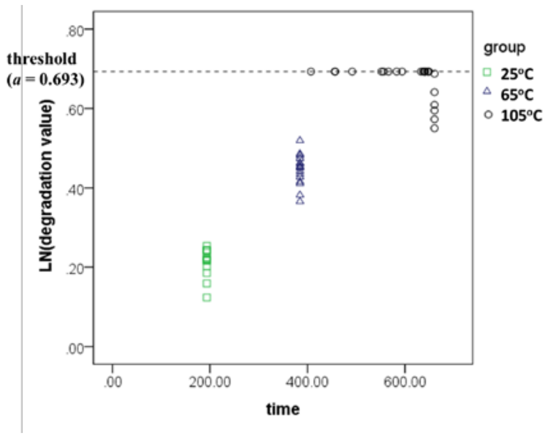

Fig. 2. The Transformed LED Data in the Form of (21).

data in (6) consists of two parts: one from the observed failure times of the failed units, and the other from the observed degradation values of the censored units. First, T_{li} independently follows $IG\left(\mu/\beta(s_l;\theta),\lambda/\beta(s_l;\theta)\right)$ for $i=1,\ldots,M_l$. Second, the $W_{li}(\alpha)$ of the ith censored unit has the following likelihood function (cf. Lee and Tang [8])

$$h_l(w_{li}(\alpha)) \equiv P(W_{li}(\alpha)=w_{li}(\alpha),\ T_{li}>\alpha)$$

$$=\sqrt{\frac{\lambda}{2\pi\alpha a^2\beta(s_l;\theta)}}e^{-\frac{\lambda\left(w_{li}(\alpha)-\frac{a\alpha\beta(s_l;\theta)}{\mu}\right)^2}{2\alpha\beta(s_l;\theta)a^2}}\left(1-e^{-\frac{2\lambda(a-w_{li}(\alpha))}{a\alpha\beta(s_l;\theta)}}\right). \quad (31)$$

Therefore, the general likelihood function of all data in (6) is

$$\prod_{l=0}^{k}\left(\prod_{i=1}^{m_l} f_l(t_{li})\times\prod_{i=m_l+1}^{n_l} h_l(w_{li}(\alpha))\right),$$

where $f_l(\cdot)$ is the pdf of $IG\left(\mu/\beta(s_l;\theta),\lambda/\beta(s_l;\theta)\right)$, and can be obtained from Chhikara and Folks [2]. The GMLEs of θ, μ, and λ (denoted $\hat{\theta}_{\text{GMLE}}$, $\hat{\mu}_{\text{GMLE}}$, and $\hat{\lambda}_{\text{GMLE}}$, respectively) are obtained by maximizing the general likelihood function above, and therefore satisfy the following equations

$$\sum_{l=0}^{k}(1/s_0-1/s_l)\left\{n_l-\hat{\lambda}_{\text{GMLE}}\left[\frac{\beta(s_l;\hat{\theta}_{\text{GMLE}})}{\hat{\mu}^2_{\text{GMLE}}}\sum_{i=1}^{M_l}T_{li}-\frac{1}{\beta(s_l;\hat{\theta}_{\text{GMLE}})}\sum_{i=1}^{M_l}\frac{1}{T_{li}}\right.\right.$$
$$+\frac{\alpha\beta(s_l;\hat{\theta}_{\text{GMLE}})}{\hat{\mu}^2_{\text{GMLE}}}(n_l-M_l)-\sum_{i=M_l+1}^{n_l}\frac{W_{li}(\alpha)^2}{\alpha\beta(s_l;\hat{\theta}_{\text{GMLE}})a^2}$$
$$\left.\left.+\sum_{i=M_l+1}^{n_l}\frac{4(a-W_{li}(\alpha))/(a\alpha\beta(s_l;\hat{\theta}_{\text{GMLE}}))}{\exp\left\{2\hat{\lambda}_{\text{GMLE}}(a-W_{li}(\alpha))/(a\alpha\beta(s_l;\hat{\theta}_{\text{GMLE}}))\right\}-1}\right]\right\}=0, \quad (32)$$

$$\hat{\mu}_{\text{GMLE}}=\frac{\sum_{l=0}^{k}\beta(s_l;\hat{\theta}_{\text{GMLE}})\left(\sum_{i=1}^{M_l}T_{li}+(n_l-M_l)\alpha\right)}{\sum_{l=0}^{k}\left(M_l+\sum_{i=M_l+1}^{n_l}W_{li}(\alpha)/a\right)}, \quad (33)$$

Table I. Estimation Results for the LED Example.

Parameter	LVE	GMLE	MMLE	MEME	MEME2
θ	0.1499	0.1655	0.1499	0.1499	0.1499
μ	618.97	682.86	618.97	618.97	618.97
λ	42676.2	40642.2	40585.9	44887.4	39214.1

$$\hat{\lambda}_{\text{GMLE}}=\left(\sum_{l=0}^{k}n_l\right)\left\{\sum_{l=0}^{k}\left[\begin{array}{l}\sum_{i=1}^{M_l}\frac{\beta(s_l;\hat{\theta}_{\text{GMLE}})(T_{li}-\hat{\mu}_{\text{GMLE}}/\beta(s_l;\hat{\theta}_{\text{GMLE}}))^2}{\hat{\mu}^2_{\text{GMLE}}T_{li}}\\ +\sum_{i=M_l+1}^{n_l}\frac{\left(W_{li}(\alpha)-a\alpha\beta(s_l;\hat{\theta}_{\text{GMLE}})/\hat{\mu}_{\text{GMLE}}\right)^2}{a^2\alpha\beta(s_l;\hat{\theta}_{\text{GMLE}})}\\ -\sum_{i=M_l+1}^{n_l}\frac{4(a-W_{li}(\alpha))/(a\alpha\beta(s_l;\hat{\theta}_{\text{GMLE}}))}{\exp\left\{2\hat{\lambda}_{\text{GMLE}}(a-W_{li}(\alpha))/a\alpha\beta(s_l;\hat{\theta}_{\text{GMLE}})\right\}-1}\end{array}\right]\right\}^{-1}. \quad (34)$$

A numerical procedure is required to iteratively obtain the GMLEs, and the results may depend on the starting values used in the procedure. For the LED example, $\hat{\theta}_{\text{GMLE}}$ = 0.1655, $\hat{\mu}_{\text{GMLE}}$ = 682.86, and $\hat{\lambda}_{\text{GMLE}}$ = 40,642.2 (Table I).

In addition to LVEs and GMLEs, we also obtain the two-stage MMLEs, MEMEs, and MEME2s of μ, and λ. To obtain these three two-stage estimates, we replace LVEs with the estimates from (9), (16), and (26) of Lee and Tang [8], respectively, in our proposed two-stage procedure. Results are also given in Table I.

Because all two-stage estimates start with the same first stage, they have the same estimate of θ. $\hat{\theta}_{\text{GMLE}}$ = 0.1655 is much larger than $\hat{\theta}_{\text{LVE}}$ = 0.1499. For estimating μ, $\hat{\mu}_{\text{GMLE}}$ = 682.86 is also much larger than $\hat{\mu}_{\text{LVE}}$ = 618.97. When estimating λ, $\hat{\lambda}_{\text{LVE}}$ = 42,676.2, $\hat{\lambda}_{\text{GMLE}}$ = 40,642.2, $\hat{\lambda}_{\text{MMLE}}$ = 40,585.9, and $\hat{\lambda}_{\text{MEME2}}$ = 39214.1 are fairly close to each other, except that $\hat{\lambda}_{\text{MEME}}$ = 44,887.4, which is expected. In the next section, we will evaluate the finite-sample performances of our proposed LVEs via a simulation study. The results show that our LVEs of all model parameters in Table I for this LED example are quite reasonable.

B. *Simulation Study*

In this study, we choose baseline values for the lifetime parameters around the estimated values in Table I: μ = 600, and λ = 40,000. Then, with a = 0.6, we obtain η = 0.001, and $\sigma=0.003$ for the degradation process (1). For the Arrhenius law, we use $\theta=0.15$. We consider two censoring times: α = 200 (short), and 320 (long). We also consider 3 lower stress levels (s_0,s_1,s_2) = (25, 40, 55), and a larger IG scale parameter

Table II. The Experimental Settings and Failure Probabilities for the Three-Level PCSADT with $\theta = 0.15$, $\mu = 600$, and λ = 40,000 (60,000)*.

Censoring Times	Stress Levels					
	25°C	40°C	55°C	25°C	65°C	105°C
a = 200	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)	(0.000) (0.000)	(0.000) (0.000)	0.859# (0.902)
a = 320	(0.000) (0.000)	0.002 (0.000)	0.223 (0.169)	(0.000) (0.000)	0.694 (0.724)	1.000# (1.000)
a = 450	0.011 (0.002)	0.488 (0.475)	0.979 (0.993)	0.011 (0.002)	0.999 (1.000)	1.000 (1.000)

λ = 60,000. Table II shows the failure probability for each scenario. By choosing the baseline values above, we have considered cases ranging from those with just a few or no failures to those with many failures. The failure probability is computed from (23).

For each scenario in Table II, we further consider four sets of sample sizes for the three stress levels of the PCSADT: $n_0 = n_1 = n_2 =$ 6 (N = 6 × 3), $n_0 = n_1 = n_2 =$ 12 (N = 12 × 3), $n_0 = n_1 = n_2 =$ 24 (N = 24 × 3), and $n_0 = n_1 = n_2 =$ 96 (N = 96 × 3), respectively. Using all baseline values, we simulate the degradation path for each test unit according to (4) to obtain data in the form of (6). We then estimate θ, μ, and λ under each method. For each case, we repeat the simulation 2,000 times, and obtain the average estimate (AvgEst), standard error (s.e.), and square root of mean square error (sqrtMSE) for each estimator. Results for α = 200, and 320, under stress levels (25, 65, 105) and $\lambda = 40{,}000$ are summarized in Tables III and IV. Results for the other 10 cases in Table II are similar, and hence given in the Supplemental Document. Our comparisons below are based on results from all cases.

Estimating θ

For estimating the life-stress parameter θ, the proposed asymptotically best linear unbiased estimates $\hat{\theta}_{\text{LVE}}$ are consistently close to the baseline value of 0.15 for all cases considered in this study. The s.e. of $\hat{\theta}_{\text{LVE}}$ is inversely proportional to the square root of the total sample size N, approximately, and hence decreases with N. On the other hand, $\hat{\theta}_{\text{GMLE}}$ consistently, and sometimes significantly, overestimates the baseline value, and increasing N does not necessarily reduce the bias of $\hat{\theta}_{\text{GMLE}}$ (bias is defined as AvgEst – baseline value). Nevertheless, its s.e. is similar to that of $\hat{\theta}_{\text{LVE}}$. We also note that a longer censoring time, or higher stress, leads to a smaller s.e. for both $\hat{\theta}_{\text{LVE}}$ and $\hat{\theta}_{\text{GMLE}}$. Overall, $\hat{\theta}_{\text{LVE}}$ is a better estimator of θ.

Estimating μ

For the mean lifetime μ, we show only the results of the comparison between $\hat{\mu}_{\text{LVE}}$ and $\hat{\mu}_{\text{GMLE}}$ because $\hat{\mu}_{\text{MEME}}$ and $\hat{\mu}_{\text{MMLE}}$ are identical to $\hat{\mu}_{\text{LVE}}$ (see our discussions in Section III). First, we observe that $\hat{\mu}_{\text{LVE}}$ consistently outperforms $\hat{\mu}_{\text{GMLE}}$ in terms of bias. In particular, $\hat{\mu}_{\text{GMLE}}$ significantly overestimates the baseline value of μ = 600, and its bias is larger when the censoring time is short (e.g., α = 200, or even 320), regardless of the stress levels. The s.e. of $\hat{\mu}_{\text{LVE}}$ and $\hat{\mu}_{\text{GMLE}}$ are similar, and have similar relationships with N as the s.e. of the two estimators for θ. We also observe clear correlations between $\hat{\theta}_{\text{GMLE}}$ and $\hat{\mu}_{\text{GMLE}}$. That is, $\hat{\mu}_{\text{GMLE}}$ with a larger bias always corresponds to a $\hat{\theta}_{\text{GMLE}}$ which also has a larger bias. This result confirms Morton's [16] comments on the biasing effect of estimating a nuisance parameter. In fact, this effect can be explained by the fact that $\hat{\theta}_{\text{GMLE}}$ affects $\hat{\beta}(s_l;\hat{\theta}_{\text{GMLE}})$ exponentially through (16), and the latter in turn affects $\hat{\mu}_{\text{GMLE}}$ linearly through (33). Overall, $\hat{\mu}_{\text{LVE}}$ is a better estimator of μ.

Estimating λ

For estimating the scale parameter λ, the proposed $\hat{\lambda}_{\text{LVE}}$ consistently estimates the baseline value (40,000, or 60,000) with a small s.e., regardless of the experimental settings. Furthermore, except for the cases with a long censoring time (e.g., α = 450) where $\hat{\lambda}_{\text{MEME2}}$ is slightly better, $\hat{\lambda}_{\text{LVE}}$ in general outperforms the other four estimators in both bias and s.e. These four estimators can have a large bias and s.e., particularly when sample sizes are small ($N = 6\times 3$, and $N = 12\times 3$). $\hat{\lambda}_{\text{LVE}}$, $\hat{\lambda}_{\text{GMLE}}$, and $\hat{\lambda}_{\text{MMLE}}$ are less sensitive to the censoring time α than $\hat{\lambda}_{\text{MEME}}$ and $\hat{\lambda}_{\text{MEME2}}$, although the latter two improve with a longer censoring time. Finally, with some derivations, we can show that, if complete failure times, say T_i, from $IG(\mu, \lambda)$ are available, the UMVUE of λ is $\hat{\lambda}_{\text{UMVUE}} = (N-3)/\sum_{i=1}^{N}(1/T_i - 1/\bar{T})$ with a standard deviation of $\sqrt{2\lambda^2/(N-5)}$. When λ = 40,000, the standard deviations are 15,689.3, 10160.0, 6,910.9, and 3,362.7 for N = 6×3, 12×3, 24×3, and 96×3, respectively. From Tables III and IV, we see that the s.e. of the proposed $\hat{\lambda}_{\text{LVE}}$ are close to the standard deviations of their corresponding UMVUEs (assuming failure times of all experimental units are available). This is also true when λ = 60,000.

In summary, our proposed LV method provides better estimators for all model parameters than the methods considered in this study in finite as well as infinite samples.

Table III. Results under α = 200, and Stress = (25°C, 65°C, 105°C).

Simulated Estimates	$\hat{\theta}_{LVE}$	$\hat{\theta}_{GMLE}$	$\hat{\mu}_{LVE}$	$\hat{\mu}_{GMLE}$	$\hat{\lambda}_{LVE}$	$\hat{\lambda}_{GMLE}$	$\hat{\lambda}_{MMLE}$	$\hat{\lambda}_{MEME}$	$\hat{\lambda}_{MEME2}$
					N=6×3				
AvgEst	0.1500	0.1602	603.22	639.10	42692.2	51563.9	51112.3	63612.4	45408.6
s.e.	0.0134	0.0129	52.26	55.92	19981.6	24817.1	24060.3	32873.7	23282.1
sqrtMSE	0.0134	0.0164	52.36	68.23	20162.1	27379.0	26502.5	40475.0	23902.1
					N=12×3				
AvgEst	0.1502	0.1608	602.94	640.64	41101.1	44437.0	44823.4	60926.7	44658.7
s.e.	0.0097	0.0095	38.16	41.06	10565.6	11592.0	11470.6	16656.3	12172.7
sqrtMSE	0.0097	0.0145	38.27	57.77	10622.8	12412.1	12443.4	26746.2	13033.8
					N=24×3				
AvgEst	0.1499	0.1610	600.98	640.08	40501.3	41686.7	42284.3	60298.2	44729.3
s.e.	0.0066	0.0064	26.00	27.99	7286.6	7654.8	7528.6	11443.5	8449.1
sqrtMSE	0.0066	0.0127	26.02	48.88	7303.9	7838.4	7867.5	23301.7	9682.6
					N=96×3				
AvgEst	0.1500	0.1613	600.03	640.31	40025.7	39777.4	40469.5	59547.9	44548.7
s.e.	0.0034	0.0034	13.29	14.30	3280.5	3376.1	3308.1	5386.2	4014.3
sqrtMSE	0.0034	0.0118	13.29	42.77	3280.6	3383.4	3341.3	20276.4	6066.7

Table IV. Results under α = 320, and Stress = (25°C, 65°C, 105°C).

Simulated Estimates	$\hat{\theta}_{LVE}$	$\hat{\theta}_{GMLE}$	$\hat{\mu}_{LVE}$	$\hat{\mu}_{GMLE}$	$\hat{\lambda}_{LVE}$	$\hat{\lambda}_{GMLE}$	$\hat{\lambda}_{MMLE}$	$\hat{\lambda}_{MEME}$	$\hat{\lambda}_{MEME2}$
					N=6×3				
AvgEst	0.1495	0.1566	598.78	621.08	42712.0	50271.9	50751.0	49990.0	42133.0
s.e.	0.0118	0.0113	41.79	42.94	17642.9	21249.7	21045.4	21552.9	18087.6
sqrtMSE	0.0118	0.0131	41.81	47.84	17850.1	23602.1	23632.5	23755.6	18212.9
					N=12×3				
AvgEst	0.1496	0.1574	599.67	624.16	40952.3	43518.1	44352.5	48182.0	41177.4
s.e.	0.0082	0.0078	29.29	29.93	10657.2	11209.0	11224.9	12581.5	10680.6
sqrtMSE	0.0082	0.0107	29.29	38.47	10699.7	11748.1	12039.2	15008.0	10745.3
					N=24×3				
AvgEst	0.1499	0.1579	599.51	624.67	40425.4	40985.7	42068.0	47710.1	41040.3
s.e.	0.0057	0.0054	19.65	20.28	6746.4	6699.3	6733.9	7964.5	6796.2
sqrtMSE	0.0057	0.0096	19.66	31.94	6759.8	6771.5	7044.3	11085.1	6875.4
					N=96×3				
AvgEst	0.1498	0.1583	600.00	626.56	40022.7	39254.0	40452.6	47379.5	40934.0
s.e.	0.0026	0.0025	9.18	9.56	3213.4	3190.4	3187.4	3952.4	3394.4
sqrtMSE	0.0026	0.0087	9.18	28.23	3213.5	3276.4	3219.4	8371.3	3520.5

VI. CONCLUSION

To estimate parameters of the lifetime distribution of manufactured products, we propose a two-stage estimation method, assuming that the failure times of failed units, and the degradation values of censored units under a time-censored PCSADT are available. We assume that the (transformed) degradation path of a test unit follows a Wiener process. The objective is to estimate the mean and scale parameter of the lifetime distribution under normal stress.

In the first stage of our estimation method, we obtain *s*-consistent estimates of the decay-acceleration factors for all stress levels. These estimates are then used to transform the available data so that the resulting data can be considered as data obtained under normal stress. In the second stage, we propose LVEs of the decay factor, μ, and λ by adding latent variables for the unobserved degradations that occur after the failure to obtain pseudo degradation values at the censoring times for the failed units. These values, along with the observed degradation values from the censored units, are then used to develop the LVEs. The proposed estimators of all model parameters are shown to be *s*-consistent and easily interpretable, and have closed-form expressions. Our simulation results show that the proposed two-stage LVEs are in general less biased, and have standard errors smaller than those from the traditional maximum likelihood method in various finite-sample scenarios. One possible direction for future research is to see whether the proposed two-stage estimation method can be applied to other types of stress tests, such as the step-stress ADT, or to other types of degradation data, such as the first-passage times of the degradation paths over certain multiple non-failure thresholds.

APPENDIX A

Proof of s-Consistency of LVE of λ in (13)

Proof: To prove the *s*-consistency, we rewrite the LVE of λ under s_0 in (13) as

$$\hat{\lambda}_n = \frac{\frac{1}{n}\left(\alpha(n-M)+\sum_{i=1}^{M}T_i\right)}{\frac{1}{n}\sum_{i=1}^{M}(1-T_i/\hat{\mu}_n)^2+\frac{1}{n}\sum_{i=M+1}^{n}(W_i/a-\alpha/\hat{\mu}_n)^2}, \quad (35)$$

where $\hat{\mu}_n$ is the LVE, or any other *s*-consistent estimator of μ. We study the weak convergence of each term in the numerator and the denominator of (35) in (*i*), (*ii*), and (*iii*), respectively, below.

(*i*) For the numerator of (35), we first note that the number of failed units M in a sample of n units follows a binomial distribution with failure probability $F(\alpha)$. Thus, $M/n \xrightarrow{P} F(\alpha)$. The conditional expectation of $\sum_{i=1}^{M}T_i/n$ in (35) is

$$\begin{aligned} E\left(\frac{1}{n}\sum_{i=1}^{M}T_i \middle| T_i\le\alpha\right) &= \frac{1}{n}E\left[E\left(\sum_{i=1}^{M}T_i \middle| T_i\le\alpha\right)\middle| M\right] \\ &= \frac{1}{n}E(M)E\left(T_1 \middle| T_1\le\alpha\right) \qquad \text{(Wald's equation)} \\ &= \frac{1}{n}nF(\alpha)\int_0^{\alpha}\left(t\frac{f(t)}{F(\alpha)}\right)dt = \int_0^{\alpha}t\, f(t)dt \\ &= \mu\left[\Phi(A)-e^{2\lambda/\mu}\Phi(-B)\right], \end{aligned} \quad (36)$$

where $\Phi(\cdot)$ is the standard normal cdf, $A=\sqrt{\lambda/\alpha}(\alpha/\mu-1)$, and $B=\sqrt{\lambda/\alpha}(\alpha/\mu+1)$. Similarly, using Wald's equation, we have

$$Var\left(\frac{1}{n}\sum_{i=1}^{M}T_i \middle| T_i\le\alpha\right) = \frac{1}{n^2}E(M)\cdot Var\left(T_1 \middle| T_1\le\alpha\right) \le \frac{1}{n}F(\alpha)\,\alpha^2,$$

which converges to 0 as $n\to\infty$. This implies that $\sum_{i=1}^{M}T_i/n$ converges in probability to its expectation in (36). Hence, the numerator in (35) is

$$\frac{\sum_{i=1}^{M}T_i+(n-M)\alpha}{n} \xrightarrow{P} \mu\left[\Phi(A)-e^{2\lambda/\mu}\Phi(-B)\right]+\alpha\left[1-\Phi(A)-e^{2\lambda/\mu}\Phi(-B)\right]. \quad (37)$$

(*ii*) For the first term in the denominator of (35), we write

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{M}\left(1-\frac{T_i}{\hat{\mu}_n}\right)^2 &= \frac{1}{n}\frac{1}{\hat{\mu}_n^2}\sum_{i=1}^{M}\left((T_i-\mu)+(\mu-\hat{\mu}_n)\right)^2 \\ &= \frac{1}{\hat{\mu}_n^2}\left[\begin{array}{l}\frac{\mu^2}{n}\sum_{i=1}^{M}\left(1-\frac{T_i}{\mu}\right)^2+2\left(\mu-\hat{\mu}_n\right)\frac{1}{n}\sum_{i=1}^{M}T_i \\ -2\mu\left(\mu-\hat{\mu}_n\right)\frac{M}{n}+\frac{M}{n}\left(\mu-\hat{\mu}_n\right)^2\end{array}\right] \end{aligned} \quad (38)$$

Because $\hat{\mu}_n \xrightarrow{P} \mu$, $0\le M/n\le 1$ with probability 1, and $0\le\sum_{i=1}^{M}T_i/n\le\alpha M/N\le\alpha$ with probability 1, the last three terms in (38) converge in probability to 0. For the first term in (38), we obtain

$$\begin{aligned} E\left(\sum_{i=1}^{M}T_i^2 \middle| T_i\le\alpha\right) &= n\int_0^{\alpha}t^2 f(t)dt \\ &= n\left\{\begin{array}{l}\mu^2\left[\Phi(A)+e^{2\lambda/\mu}\Phi(-B)\right]+\frac{\mu^3}{\lambda}\left[\Phi(A)-e^{2\lambda/\mu}\Phi(-B)\right] \\ -\frac{\mu^3}{\lambda}\left[(A+B)\phi(A)\right]\end{array}\right\}, \end{aligned}$$

and hence

$$\begin{aligned} &E\left[\frac{1}{n}\sum_{i=1}^{M}(1-T_i/\mu)^2 \middle| T_i\le\alpha\right] \\ &= \frac{1}{n}\left[\mu^{-2}E\left(\sum_{i=1}^{M}T_i^2\,|\,T_i\le\alpha\right)-2\mu^{-1}E\left(\sum_{i=1}^{M}T_i\,|\,T_i\le\alpha\right)+E(M)\right] \\ &= \frac{\mu}{\lambda}\left[\Phi(A)-e^{2\lambda/\mu}\Phi(-B)\right]+4e^{2\lambda/\mu}\Phi(-B)-\frac{\mu}{\lambda}(A+B)\phi(A) \\ &= \frac{\mu}{\lambda}\left[\Phi(A)-e^{2\lambda/\mu}\Phi(-B)\right]+4e^{2\lambda/\mu}\Phi(-B)-2\sqrt{\frac{\alpha}{\lambda}}\phi(A), \end{aligned} \quad (39)$$

where $\phi(\cdot)$ is the standard normal pdf. Now, because

$$\begin{aligned} &Var\left(\frac{1}{n}\sum_{i=1}^{M}(1-T_i/\mu)^2\,|\,T_i\le\alpha\right) \\ &= \frac{1}{n^2}E(M)\cdot Var\left((1-T_1/\mu)^2\,|\,T_1\le\alpha\right) \\ &\le \frac{1}{n}F(\alpha)\left(1+\frac{\alpha}{\mu}\right)^4 \to 0, \end{aligned}$$

as $n\to\infty$, $n^{-1}\sum_{i=1}^{M}(1-T_i/\mu)^2$ in (38) converges in probability to its expectation in (39), thus, the first term in the denominator of (35) converges in probability to (39).

(*iii*) For the second term in the denominator of (35), we note that

$$\begin{aligned} \frac{1}{n}\sum_{i=M+1}^{n}\left(\frac{W_i}{a}-\frac{\alpha}{\hat{\mu}_n}\right)^2 &= \frac{1}{n}\sum_{i=M+1}^{n}\left(\left(\frac{W_i}{a}-\frac{\alpha}{\mu}\right)+\left(\frac{\alpha}{\mu}-\frac{\alpha}{\hat{\mu}_n}\right)\right)^2 \\ &= \frac{1}{n}\sum_{i=M+1}^{n}\left(\frac{W_i}{a}-\frac{\alpha}{\mu}\right)^2+2\left(\frac{\alpha}{\mu}-\frac{\alpha}{\hat{\mu}_n}\right)\frac{1}{n}\sum_{i=M+1}^{n}\left(\frac{W_i}{a}-\frac{\alpha}{\mu}\right) \\ &\quad+\frac{n-M}{n}\left(\frac{\alpha}{\mu}-\frac{\alpha}{\hat{\mu}_n}\right)^2 \end{aligned} \quad (40)$$

Again, because $\hat{\mu}_n \xrightarrow{P} \mu$ and $M/n \xrightarrow{p} F(\alpha)$, the third term in (40) converges in probability to 0. Furthermore, because all W_i have a finite conditional variance, the conditional variance of $\sum_{i=M+1}^{n} W_i/n$ converges to 0 as $n \to \infty$. Consequently, $\sum_{i=M+1}^{n} W_i/n$ will converge in probability to its expectation

$$E\left(\frac{1}{n}\sum_{i=M+1}^{n} W_i \mid T_i > \alpha\right) = \frac{1}{n}E(n-M)\cdot E\left(W \middle| T > \alpha\right)$$
$$= \frac{1}{n} n(1-F(\alpha))\int_{-\infty}^{a} w\frac{h(w)}{1-F(\alpha)}dw = \int_{-\infty}^{a} w h(w)dw,$$

where $h(\cdot)$ is the joint likelihood of $W(\alpha)$ given in (31) with $\beta(s_l;\theta)=1$. It is understood that (W,T) above is i.i.d. as (W_i,T_i) for $i = M+1,...,n$. So, the second term in (40) converges in probability to 0. For the first term in (40),

$$E\left[\frac{1}{n}\sum_{i=M+1}^{n}\left(\frac{W_i}{a}-\frac{\alpha}{\mu}\right)^2 \mid T_i > \alpha\right] = \int_0^a \left(\frac{w}{a}-\frac{\alpha}{\mu}\right)^2 h(w)dw$$
$$= \frac{\alpha}{\lambda}\left[1-\Phi(A)-e^{2\lambda/\mu}\Phi(-B)\right] + \frac{\alpha}{\lambda}(A-B)\phi(A)$$
$$+4\sqrt{\frac{\alpha}{\lambda}}\phi(A) - 4e^{2\lambda/\mu}\Phi(-B)$$
$$= \frac{\alpha}{\lambda}\left[1-\Phi(A)-e^{2\lambda/\mu}\Phi(-B)\right] + 2\sqrt{\frac{\alpha}{\lambda}}\phi(A) - 4e^{2\lambda/\mu}\Phi(-B). \quad (41)$$

Furthermore, the conditional variance is

$$Var\left[\frac{1}{n}\sum_{i=M+1}^{n}\left(\frac{W_i}{a}-\frac{\alpha}{\mu}\right)^2 \mid T_i > \alpha\right]$$
$$= \frac{1}{n^2}E(n-M)\cdot Var\left[\left(\frac{W}{a}-\frac{\alpha}{\mu}\right)^2 \mid T > \alpha\right]$$
$$\le \frac{1}{n}\int_{-\infty}^{\alpha}\left(\frac{w}{a}-\frac{\alpha}{\mu}\right)^4 h(w)dw$$
$$\le \frac{1}{n}\int_{-\infty}^{\infty}\left(\frac{w}{a}-\frac{\alpha}{\mu}\right)^4 g(w)dw \to 0, \text{ as } n\to\infty,$$

where

$$g(w) \equiv \frac{1}{\sqrt{2\pi\alpha a^2/\lambda}}\exp\left\{-\lambda\left(w-a\alpha/\mu\right)^2 / (2\alpha a^2)\right\} > h(w),$$

and is proportional to a normal pdf. Thus, the first term in (40) also converges in probability to its expectation in (41). Hence, the denominator of (35) is

$$\frac{1}{n}\left[\sum_{i=1}^{M}(1-T_i/\mu)^2 + \sum_{i=M+1}^{n}(W_i/a-\alpha/\mu)^2\right]$$

$$\xrightarrow{P} \frac{\mu}{\lambda}\left[\Phi(A)-e^{2\lambda/\mu}\Phi(-B)\right] + \frac{\alpha}{\lambda}\left[1-\Phi(A)-e^{2\lambda/\mu}\Phi(-B)\right]. \quad (42)$$

Finally, from (37), (42), and the Slusky's theorem, we prove

$$\hat{\lambda}_n = \frac{\frac{1}{n}\left(\alpha(n-M)+\sum_{i=1}^{M}T_i\right)}{\frac{1}{n}\left(\sum_{i=1}^{M}(1-T_i/\hat{\mu}_n)^2 + \sum_{i=M+1}^{n}(W_i/a-\alpha/\hat{\mu}_n)^2\right)} \xrightarrow{P} \lambda. \quad \blacksquare$$

Proofs of s-Consistencies of $\hat{\mu}_{\text{LVE}}$ in (28), and $\hat{\lambda}_{\text{LVE}}$ in (30)

Proof. To prove that $\hat{\mu}_{\text{LVE}}$ is *s*-consistent for μ, we first note that

$$\hat{\mu}_{\text{LVE}} = \frac{\sum_{l=0}^{k}\hat{\beta}_l\left((n_l-M_l)\alpha+\sum_{i=1}^{M_l}T_{lj}\right)}{\sum_{l=0}^{k}\left(\sum_{i=M_l+1}^{n_l}W_{li}(\alpha)/a+M_l\right)}$$
$$= \sum_{l=0}^{k}\left[\frac{\sum_{i=M_l+1}^{n_l}W_{li}(\alpha)/a+M_l}{\sum_{l=0}^{k}\left(\sum_{i=M_l+1}^{n_l}W_{li}(\alpha)/a+M_l\right)}\right]\left(\hat{\beta}_l\frac{((n_l-M_l)\alpha+\sum_{i=1}^{M_l}T_{lj})}{\sum_{i=M_l+1}^{n_l}W_{li}(\alpha)/a+M_l}\right)$$
$$= \sum_{l=0}^{k}\left[\frac{\sum_{i=M_l+1}^{n_l}W_{li}(\alpha)/a+M_l}{\sum_{l=0}^{k}\left(\sum_{i=M_l+1}^{n_l}W_{li}(\alpha)/a+M_l\right)}\right]\hat{\beta}_l\frac{a}{\hat{\eta}_l}. \quad \text{(by (14))} \quad (43)$$

From $\hat{\beta}_l \xrightarrow{p} \beta_l$ and $a/\hat{\eta}_l \xrightarrow{p} a/\eta_l$, we have $\hat{\beta}_l(a/\hat{\eta}_l) \xrightarrow{P} \beta_l(a/\eta_l) = \mu$ by (15) and (3). Because the weights in (43) always sum to 1, it is straightforward to show $\hat{\mu}_{\text{LVE}} \xrightarrow{p} \mu$.

The proof of *s*-consistency of $\hat{\lambda}_{\text{LVE}}$ in (30) is similar to the earlier proof for the LVE in (13), and therefore is briefly described below. First, we rewrite (30) as (44). Then, using the fact that $\beta_l T_{li}$ independently follow $IG(\mu,\lambda)$, $\hat{\beta}_l \xrightarrow{P} \beta_l$, $\hat{\mu}_{\text{LVE}} \xrightarrow{P} \mu$, and $W_{li}(\alpha)$ is a standard Wiener process at time $\beta_l\alpha$, we can show that the numerator of (44) converges in probability to (cf. (37))

$$\sum_{l=0}^{k}\frac{1}{\beta_l}\pi_l\left\{\mu\left[\Phi(A_l)-e^{2\lambda/\mu}\Phi(-B_l)\right]+\tilde{\alpha}_l\left[1-\Phi(A_l)-e^{2\lambda/\mu}\Phi(-B_l)\right]\right\}, \quad (45)$$

$$\hat{\lambda}_{\text{LVE}} = \frac{\sum_{l=0}^{k}\frac{1}{\beta_l}\frac{n_l}{N}\left[\frac{1}{n_l}\left(\sum_{i=1}^{M_l}\beta_l T_{li}+(n_l-M_l)\beta_l\alpha\right)\right]}{\sum_{l=0}^{k}\frac{1}{\hat{\beta}_l}\frac{n_l}{N}\left[\frac{1}{n_l}\sum_{i=1}^{M_l}\left(1-\hat{\beta}_l T_{li}/\hat{\mu}_{\text{LVE}}\right)^2+\frac{1}{n_l}\sum_{i=M_l+1}^{n_l}\left(W_{li}(\alpha)/a-\hat{\beta}_l\alpha/\hat{\mu}_{\text{LVE}}\right)^2\right]} \quad (44)$$

where $n_l/N \to \pi_l$ as $N \to \infty$, A_l and B_l are obtained from *A* and *B*, respectively, by replacing α with $\tilde{\alpha}_l$. Similarly, the denominator of (44) converges in probability to (cf. (42))

$$\sum_{l=0}^{k} \frac{1}{\beta_l}\pi_l \left\{ \frac{\mu}{\lambda}\left[\Phi(A_l) - e^{2\lambda/\mu}\Phi(-B_l)\right] + \frac{\tilde{\alpha}_l}{\lambda}\left[1 - \Phi(A_l) - e^{2\lambda/\mu}\Phi(-B_l)\right] \right\}. \quad (46)$$

Finally, $\hat{\lambda}_{\mathrm{LVE}}$ converges in probability to (45)/(46), which is λ. Hence, we prove the *s*-consistency of $\hat{\lambda}_{\mathrm{LVE}}$. ∎

APPENDIX B

For cases where multiple stress factors are included in the analysis, the technique in this paper can be extended with only some minor modifications in the notation and a generalization of the Arrhenius law. Consider the 2-factor case, for example. We will put all factors in a factor vector, $\underline{s}_l \equiv (s_{1l}\ s_{2l})'$ and use the following Generalized Eyring Model (GEM), of which the univariate Arrhenius law, the power law, and the exponential law model are all special cases (Meeker and Escobar [15]):

$$\beta_l \equiv \beta(\underline{s}_l, \underline{\theta}) = \exp(\theta_0 + \theta_1 g_1(s_{1l}) + \theta_2 g_2(s_{2l}) + \theta_{12} g_1(s_{1l}) g_2(s_{2l})), \quad \text{for } l = 1, \ldots, k,$$

where $\underline{\theta} = (\theta_0, \theta_1, \theta_2, \theta_{12})'$; g_1 and g_2 are some functions. In matrix form, we have

$$\ln \underline{\beta} = \mathrm{A}\underline{\theta},$$

where

$$\ln \underline{\beta} \equiv \begin{pmatrix} \ln\beta_1 \\ \vdots \\ \ln\beta_k \end{pmatrix}_{k\times 1}, \quad \underline{\theta} \equiv \begin{pmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_{12} \end{pmatrix}_{4\times 1},$$

$$\mathrm{A} \equiv \begin{pmatrix} 1 & g_1(s_{11}) & g_2(s_{21}) & g_1(s_{11})\cdot g_2(s_{21}) \\ \ldots & \ldots & \ldots & \ldots \\ 1 & g_1(s_{1k}) & g_2(s_{2k}) & g_1(s_{1k})\cdot g_2(s_{2k}) \end{pmatrix}_{k\times 4},$$

and A is called the design matrix. Since we have data on β_l, namely $\hat{\beta}_l \equiv \hat{\beta}(s_l)$ in (15), the model for estimating $\underline{\theta}$ is

$$\ln \hat{\underline{\beta}} = \mathrm{A}\underline{\theta} + \underline{\varepsilon},$$

where $\underline{\varepsilon}$ is an error vector with the covariance matrix $\Sigma \equiv \mathrm{cov}(\ln \hat{\underline{\beta}})$. Note that $\ln\hat{\beta}_l$ are correlated with unequal variances: $Var(\ln\hat{\beta}_l) = \delta_0^2 + \delta_l^2$ and $Cov(\ln\hat{\beta}_l, \ln\hat{\beta}_{l'}) = \delta_0^2$ for $1 \le l, l' \le k$. Let $\Sigma^{-1/2}$ be a symmetrical matrix such that $\Sigma^{-1/2}\Sigma\Sigma^{-1/2} = \mathrm{I}$, then the Best Linear Unbiased Estimator (BLUE) of $\underline{\theta}$ is (see Seber and Lee [22]):

$$\hat{\underline{\theta}} = \left(\mathrm{A}'\Sigma^{-1}\mathrm{A}\right)^{-1}\mathrm{A}'\Sigma^{-1}\ln\hat{\underline{\beta}}.$$

When $k = 2$ with $\theta_0 = 0$ (hence, there is no need to estimate), it can be verified that the $\hat{\underline{\theta}}$ obtained above is identical to the $\hat{\theta}_{\mathrm{LVE}}$ given in (18). In other words, our LVE of θ is BLUE.

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