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PREPAYMENTS AND THE VALUATION OF ADJUSTABLE RATE MORTGAGE-BACKED SECURITIES

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The Federal National Mortgage Association (Fannie Mae) issued the first Adjustable Rate Mortgage (ARM)-Backed Security in January 1984, and the Federal Home Loan Mortgage Corporation (Freddie Mac) followed suit in April 1986. Both programs have experienced reasonable success and an active secondary market for the trading of ARM-backed securities has evolved.

The two programs are structured along similar lines. Private mortgage lenders originate adjustable rate mortgage loans; if the loans meet certain criteria, they can be pooled to support the issuance of a Fannie Mae or Freddie Mac ARM-backed security. Because the authorizing agency guarantees cash payments to the holder, the securities are considered (nearly) default-free from the perspective of the investor.

Under both programs interest payments are tied to an interest rate index. If interest payments were adjusted instantaneously, and in the absence of other frictions, the value of ARM-backed securities would remain fixed. In particular, if the securities are issued at par, they should always trade at face value. In fact, under both programs interest payments are subject to both annual and lifetime caps as well as annual, rather than instantaneous, adjustment. Both the interest rate caps and the lag in the interest rate adjustment process have the effect of pulling the price of the security away from par. Or, to put it another way, both the caps and the adjustment lag induce variability in the price of ARM-backed securities.

Offsetting the effect of the caps and the adjustment lag is the propensity of mortgagors to pay off loans before maturity. When a mortgagor pays off a loan prior to maturity, the investor receives the full face value of the loan. Mortgage prepayments tend to push the value of ARM-

backed securities toward par value and thus play an important role in the valuation of the securities.

This article explores the role of mortgage prepayments in the valuation of ARM-backed securities. We focus in particular on Freddie Mac ARM-backed securities. A comprehensive data base of Freddie Mac Series 35 ARM-backed securities issued through February 1990 is used to estimate an empirical prepayment function. The estimated prepayment function is then incorporated into the Brennan-Schwartz [1982] two-factor model of the term structure of interest rates to develop an ARM-backed securities valuation model that takes into account interest rate caps, interest rate adjustment lags, and mortgage prepayments¹.

The primary result is that, in the absence of prepayments, ARM-backed securities are subject to wide swings in value in response to changes in interest rates, whereas when prepayments are considered, the effect of fluctuations in value are substantially dampened. This result underscores the importance of considering prepayments in ARM-backed security valuation.

ARM-BACKED SECURITIES

There are a variety of Freddie Mac and Fannie Mae programs under which adjustable rate mortgages can be used to support the issuance of ARM-backed securities. In terms of secondary market trading, by far the most popular Freddie Mac program is the Series 35. Under this program, Freddie Mac purchases adjustable rate mortgages from mortgage originators and groups them into pools that are then used to support the issuance of ARM-backed securities. In return for a fee, Freddie Mac guarantees the timely payment of interest and principal to the holders of the securities. That is, Freddie Mac insures the investor against loss due to delinquency and/or default by mortgagors.

The characteristic distinguishing the various Freddie Mac and Fannie Mae programs is the type of collateral that can be used to support the ARM-backed security. Across the various programs, the collateral differs according to the frequency of interest rate adjustment (semiannually, annually, or triannually), the index to which interest payments are tied, the annual interest rate adjustment cap, and the lifetime interest rate cap. Underlying mortgages in the Series 35 program all have an annual interest rate adjustment interval, are tied to the one-year U.S. Treasury index, and have an annual interest rate adjustment cap of $\pm 2\%$.

Each mortgage that backs a security has an initial coupon rate of interest. Typically, this initial rate is less than the "fully indexed" rate (which is equal to the index rate

plus a specified margin). Often called the "teaser rate," the initial rate is usually in effect during the first six months of the loan's life (the "teaser period"). Once the teaser period ends, the interest rate of the loan is set to its fully indexed level.

All loans in a specific pool need not have the same teaser rate or the same margin. At the end of each twelve-month period following the end of the teaser period, the coupon rate of interest is reset according to the level of the one-year Treasury index. Each loan is fully amortizing over a specified period, which may not exceed 360 months, however. Mortgages pooled to issue a security may have any remaining term-to-maturity. Although each mortgage also has a life-of-loan cap, the individual mortgages within a specific pool can have different caps.

Thus, under the Series 35 program (and most other Fannie Mae and Freddie Mac programs) the underlying collateral for an ARM-backed security can be heterogeneous with respect to remaining term-to-maturity, coupon interest rate, life-of-loan cap, interest rate margin relative to the index, teaser rate, and time remaining in the teaser period.

The coupon rate of interest on the ARM-backed security itself is also specified as a fixed margin over the one-year Treasury index. It adjusts annually and is subject to a $\pm 2\%$ annual adjustment cap and a specified lifetime cap. The margin and lifetime caps of the security are lower than the weighted averages of those of the adjustable rate mortgages that underlie the security. Promised monthly principal and interest payments along with any unscheduled principal prepayments are passed through to investors on a pro rata basis after the Freddie Mac guarantee fee and the mortgage originator's servicing fee have been subtracted.

ARM-BACKED SECURITY PREPAYMENTS

Theory

Valuation of ARM-backed securities must take into account the index to which the monthly interest payments are linked, the margin over the index, the periodic interest rate reset interval, and the annual adjustment and lifetime interest rate caps. Because the underlying mortgages are fully callable at any time without penalty, valuation must also account for prepayments by mortgagors. In general, the choice of independent variables used to estimate a prepayment function is contingent upon the valuation procedure employed. The procedure that we use is based on the Brennan and Schwartz [1979] two-factor model of the term structure. In this framework, prepayments can be specified

as a function of the characteristics of the security, the short- and long-term interest rates, the history of interest rates, and time.

For purposes of estimating a prepayment function we conceptually classify individuals who prepay their adjustable rate mortgages as falling into one of three categories: 1) relocators; 2) refinancers; and 3) switchers. At the outset of the loan, a mortgagor chooses between a fixed rate loan and an adjustable rate loan. Dunn and Spatt [1988] argue that, because fixed rate loans embed a long-term call option, individuals who are likely to sell their houses to relocate in the near future tend to select an adjustable rate loan. This argument predicts that adjustable rate mortgages will experience higher prepayment rates in the early years of the loan than otherwise comparable fixed rate mortgages.

Flannagan, Herskovitz, and Loy [1989] provide evidence on this point. Comparison of prepayments on Fannie Mae ARM-backed securities with prepayments on otherwise similar Fannie Mae fixed rate mortgage-backed securities indicates that ARM prepayments are higher than fixed rate prepayments throughout the first five years after issuance. (Their data cover five years.) They also document a "seasoning effect" in ARM-backed security prepayments in which the rate of prepayments increases until approximately the twenty-fourth month after origination and then levels off. In the same vein, because most prepayments due to relocation take place during the summer months, adjustable rate mortgages are likely to experience a stronger seasonal effect than do otherwise similar fixed rate mortgages.

In the empirical analysis that follows, we include two variables to capture the effects of relocators on prepayments. The first is a time variable, which measures the length of time (or, more precisely, the weighted average number of months) that the mortgages supporting the ARM-backed security have been outstanding. The second is a dummy variable to capture seasonality in ARM prepayments.

Refinancers are individuals who select an adjustable rate mortgage at the outset because they prefer an adjustable rate loan to a fixed rate loan. However, because there is a lag in the interest rate adjustment, if the index rate drops sharply during the interval between interest rate reset dates, these individuals will choose to prepay their current adjustable rate loan to take out another adjustable rate loan linked to the new lower index rate. Our empirical analysis includes changes in the short-term interest rate to capture the effects of refinancing on ARM-backed security prepayments.

Additionally, refinancing by adjustable rate mortgagors may be affected by a "burnout" factor much as refinancers are in the case of fixed rate loans. Richard and Roll [1989] report that prepayments on fixed rate loans are a function of the history of interest rates. They argue that fixed rate mortgagors are differentially sensitive to declines in the rate on fixed rate mortgages when making refinancing decisions. The first time that the market coupon rate on fixed rate mortgages falls below the coupon rate of an existing mortgage, for example, the most sensitive mortgagors in a pool will refinance. That is, the most rate-sensitive fixed rate mortgage refinancers will "burn out" of the pool. The second time that the pool is subject to a decline in the current market rate to this same level, prepayments will be lower than during the first interest rate cycle. Only if the current rate on fixed rate mortgages falls below its previous low will the next level of rate-sensitive fixed rate refinancers be induced to refinance their loans.

By extension, a similar effect may be present with refinancers in ARM-backed securities. The first time the index rate declines, some adjustable rate mortgagors will be induced to refinance to another adjustable rate loan. Others within the same pool may not refinance, however, until the rate has fallen to an even lower level. To capture the possible "burnout" effect in adjustable rate loan refinancers, we include the history of the short-term rate in the empirical estimation of the prepayment function.

Switchers are individuals who prefer a fixed rate loan, but, because of circumstances that they perceive as temporary at the outset of the loan, choose an adjustable rate mortgage, with the expectation of switching to a fixed rate loan when economic circumstances change. Characterizing the prepayment behavior of switchers is difficult because their initial choice depends upon their expectations regarding the future. Ex post, virtually any prepayment pattern can be attributed to changes in expectations about the future.

Assume, though, that switchers prefer a fixed rate loan because they prefer to smooth their future cash outflows, that their mortgage payment is a significant fraction of their budget, and that a fixed rate loan is the least costly way to smooth future cash outflows. This does not mean that either fixed or adjustable rate loans are mispriced. Rather, it means that cash outflows under an adjustable rate loan are more variable than under a fixed rate loan, and that the costs of entering into other financial contracts (such as options and futures) to offset this variability in cash flows (i.e., the cost of smoothing the cash outflows) are greater than the costs of entering into a long-term, fixed rate mortgage.

For switchers, changes in both the relative and absolute level of interest rates could be important determinants of their prepayment behavior. First, consider changes in the relative levels of the short- and long-term rate. Assume that switchers view the long-term rate as an unbiased predictor of the future short-term rate to which their interest payments are indexed. If the slope of the yield curve declines (i.e., the long rate declines relative to the short rate), switchers will perceive that the future short-term rate will be low relative to its previously expected level. In that case, prepayments due to switching will be positively correlated with changes in the slope of the yield curve. If the slope declines, ARM-backed security prepayments will decline.

Again, we should emphasize that switchers prefer fixed rate loans and do not interpret the decline in the long-term rate relative to the short-term rate to indicate that loans are mispriced. Rather, they view their current adjustable rate loan as more desirable than a fixed rate loan, given the prediction embedded in the yield curve that the index rate in the future will be lower than previously expected. Thus, relative to their decision at the origination of their adjustable rate loan, selection of a fixed rate loan is now (i.e., after the slope of the yield curve declines) even less attractive.

Ironically, this implies that increases in the slope of the yield curve will increase prepayments because of switching. That is, switchers are more inclined to switch to a fixed rate loan when the slope of the yield curve increases because the prediction embedded in the yield curve is that the short-term index rate in the future will be higher than previously expected. In our empirical estimation of the prepayment function, we use the change in the spread between the long-term rate and the short-term rate to capture changes in the slope of the yield curve.

Predictions about the relation between changes in the absolute level of interest rates and prepayments because of switching are more difficult. Switchers have chosen an adjustable rate loan only temporarily, with the expectation of switching to a fixed rate loan as soon as feasible. Some mortgagors may make this initial choice with the belief that they can predict the future coupon rate of fixed rate mortgages. Furthermore, at the time they make their choice, their prediction is that the fixed rate will decline to a lower level in the future. Presumably, when the fixed rate hits their predicted level, those mortgagors who have predicted this as the lowest level to which the fixed rate will decline switch to a fixed rate loan.

Thus, each time the rate on fixed rate loans hits a

new lower level, some additional adjustable rate mortgagors will be induced to prepay and switch to a fixed rate loan. In the empirical analysis, we use the long-term rate as a proxy for the fixed mortgage rate, adding an indicator variable to identify each occasion on which the long-term rate hits a new minimum level following the origination of the pool.

ARM-backed security prepayments are also likely to depend upon the margins of the loans underlying the security. Once mortgagors decide to borrow by means of an adjustable rate mortgage, they choose from a menu of possible terms. A lower margin, for example, can be traded off against higher upfront "points." This trade-off can induce a self-selection bias in which mortgagors who expect to switch to a fixed rate loan in the near future will exhibit a greater propensity to choose loans with higher margins at the outset.²

If so, the empirical prediction is that pools with higher weighted average margins will experience a higher rate of prepayments than pools with lower weighted average margins. In the empirical estimation, the difference between the weighted average margin of the loans in a specific pool and the sample average margin across all pools is used to capture the potential predictive ability of the margin on switching behavior.

In the same vein, the mortgagor can trade off lower points against a higher lifetime cap. As with the margin, a self-selection bias may occur in that mortgagors who expect to pay off their loans in the near future select a loan with a higher lifetime cap in exchange for lower points up front. If so, the prediction is that pools with higher weighted average lifetime caps will experience a higher rate of prepayments. In the empirical analysis, we use the difference between the weighted average lifetime cap of the loans in a specific pool and the sample average lifetime cap across all pools to capture the effect of the lifetime cap on prepayment behavior.

To summarize, logic and previous research suggest that, because of relocators, prepayments on ARM-backed securities will be dependent upon the season of the year and the length of time since origination of the pool; because of refinancers, prepayments will be negatively correlated with changes in the short-term interest rate; and because of switchers, prepayments will be positively correlated with changes in the slope of the yield curve, positively correlated with the spread between the average margin of the loans within a pool and the average margin across all pools in the sample, and positively correlated with the spread between the average lifetime cap of the loans within a pool and the average lifetime cap across all pools. Because of the burnout of refinancers, prepayments will depend

upon the history of the short-term interest rate, and because some mortgagors attempt to predict the future rate on fixed rate mortgages, prepayments will depend upon the level of the long-term rate relative to its level at the initiation of the pool.

Data

The prepayment data cover the period from April 1986 through February 1990 and represent all ARM-backed securities issued under the Series 35 program through February of 1990 — 116 pools in all. For the pool outstanding for the longest time, the maximum number of months of available prepayment data is forty-seven. For each pool, monthly observations of the expected principal balance based on the scheduled amortization and the actual observed balance are provided.

Because of mortgage prepayments, the observed balance is smaller each month than the expected balance. The time series of these two balances is used to determine the dollar amount of prepayments each month. That is, the difference between the scheduled principal payment and the actual principal payment is our estimate of the dollar amount of prepayments.

Each pool consists of mortgages with a range of remaining terms-to-maturity and a range of coupon interest rates. For each pool, the data base provides the range of remaining terms-to-maturity of the individual mortgages in the pool, as of the issuance date of the security, along with the weighted average remaining term-to-maturity. Likewise, the ranges and weighted averages of the coupon rates of interest on the underlying mortgages, their lifetime caps, and their margins over the one-year Treasury index are given for each pool, as of the issuance date of the security. As we have noted, there is a cap on the annual change in the interest rate of $\pm 2\%$ for all mortgages.

At issuance, the weighted average maturities of the pools range from 346 months to 359 months. Weighted average coupons on the pools at origination range from 7.13% to 10.5%; the lifetime caps range from 13.13% to 16.75%; and the weighted average margins over the one-year Treasury index range from 2.31% to 3.35%. These are the weighted averages of the adjustable rate mortgages that collateralize the ARM-backed securities.

Additionally, for each ARM-backed security, the data base provides the coupon rate of interest, the margin over the one-year Treasury rate, and the lifetime interest rate cap of the security itself. Because we are interested in the prepayment behavior of the underlying mortgages, the characteristics of the ARM-backed securities are not rele-

vant for estimation of our prepayment function, although they are relevant for developing the valuation model. For our sample, the lifetime caps of the ARM-backed securities range from 12.25% to 15.75%, and the margins range from 1.5% to 2.25%. The margins and lifetime caps of the securities are less than those of the underlying loans.

Monthly observations of interest rates are obtained from the *Federal Reserve Bulletin*. The three-month annualized Treasury bill rate is used as a proxy for the short-term interest rate, and the thirty-year Treasury bond rate is used as a proxy for the long-term interest rate. The index used for the Freddie Mac Series 35 ARMs is the one-year Treasury index. It is published weekly by the Board of Governors of the Federal Reserve System (in Federal Reserve Statistical Release H.15) as estimated from the Treasury's daily yield curve. The weekly index is computed as an average of the daily imputed values. Thus there are four or five values of the index calculated each month. The rate for the week just prior to the fifteenth of each month is chosen as the index rate for that month.

Estimation Procedure

Following Green and Shoven [1986], Quigley [1987], and Schwartz and Torous [1989], we use a proportional hazards model to estimate our prepayment function. Green and Shoven [1986] and Quigley [1987] estimate a prepayment model with data for individual conventional whole loans, and Schwartz and Torous [1989] extend this analysis to estimate a model for pools of fixed rate mortgages that collateralize fixed rate mortgage-backed securities.

The proportional hazards model was developed originally for the statistical analysis of failure time data. The procedure is used, for example, for estimating the probability distribution functions of failure over time of machine components or the mortality rate of biological organisms. In our application, the prepayment of a mortgage is treated as a failure event. Time is assumed to be a major variable determining prepayments along with a set of other covariates. The model can be estimated using maximum likelihood estimation or partial likelihood estimation.

Schwartz and Torous [1989] assume a log-logistic function for their baseline hazards rate and conduct a maximum likelihood estimation for their prepayment function. The virtue of the partial likelihood estimation of the proportional hazards model is that the effect of several explanatory variables can be studied before the effect of time on prepayments is considered. As the functional dependence of prepayments on time is not known ex ante, the partial likelihood procedure facilitates a more accurate estimation of

the model parameters. Hence we use that estimation procedure. A detailed derivation of the likelihood function is shown in the appendix.

The proportional hazards representation of the conditional probability of prepayment is

$$\pi(t, \nu, \theta) = \pi_0(t) \phi(\mathbf{x}, \beta). \quad (1)$$

where $\pi(t, \nu, \theta)$ is the conditional probability of prepayment of a mortgage at time t , assuming the loan has not been prepaid before then, and boldface type indicates a vector. The function $\pi_0(t)$ is the baseline hazards function, which is the prepayment rate in the absence of the effects of the explanatory variables other than time.

The vectors \mathbf{x} and β represent the exogenous variables (other than time) that determine prepayments and their associated parameters, respectively. In our model, the exponential form, $\exp(\mathbf{x}\beta)$, is used as the functional form for $\phi(\mathbf{x}, \beta)$.

Estimation of the prepayment function involves three steps. First, the parameters β are estimated as in Kalbfleisch and Prentice [1980]. In the second step, the estimated values of β are substituted into the function $\exp(\mathbf{x}\beta)$ which is used in conjunction with the observations of the covariates \mathbf{x} to determine the values of the baseline prepayment rate, $\pi_0(t)$, over time, where time is measured as the weighted average number of months since origination of the mortgages in the pool. Third, a smooth curve is fitted over the points obtained from the baseline prepayment rate to obtain an estimate of a continuous function for $\pi_0(t)$. This function captures the effect of aging on mortgage prepayments in the absence of the effects of the other covariates.

Ideally, to estimate a proportional hazards prepayment model, the actual number of mortgages prepaying in each month would be used. Only the dollar amount of prepayments is provided for the Freddie Mac data, however. An approximate procedure is to consider all the mortgages in each pool to be of the same size and translate the dollar amounts of prepayments into the number of mortgages prepaid. As do Schwartz and Torous (S&T) [1989], we assume that each mortgage has a face value of \$100,000 at origination.³ The dollar amount of prepayments is divided by \$100,000 to estimate the number (or rate) of prepayments each month. These are the observations of the dependent variable in our estimation.

Model Estimation

A set of seven covariates is chosen to estimate the

prepayment function. The first covariate is used to capture the seasonal variations in prepayment rates. Based on the assumption that relocators are more likely to move during the spring and summer months, a seasonal indicator variable, x_s , is assigned a value of 1 for the months April through September and a value of zero for other months.

For most mortgagors, there is a lag between the decision to pay off the loan and the actual cash flows to the holder of the ARM-backed security. The lag occurs for two reasons. First, for both refinancers and switchers, a new loan application must be processed and approved, which takes time. Second, there is a lag between the time the loan is paid off and the time that the payment is passed through to investors. We observe the cash flows only when they are passed through to investors. For these reasons, in estimating the prepayment model changes in interest rates are lagged relative to the month in which the prepayment is observed.

The relative short-term interest rate is used to capture the effect of refinancers on ARM-backed security prepayments. The relative change in the short-term interest rate is defined as $x_r = [\{r(t-k) - r(t-k-q)\} / r(t-k-q)]$, where r is the three-month Treasury bill rate in month $t-k$. Month t is the month in which the prepayment is observed, k is the number of months the interest rates are lagged, and q is the number of months since the last previous interest rate adjustment month.

Ideally, we would compare the short-term rate on each adjustment date with the short-term rate over the ensuing twelve months, but all the loans in a given pool do not have the same interest rate adjustment date. Thus, within a pool, the time period from the current month to the last previous adjustment date may vary from one to twelve months. In the empirical analysis, we experimented with values of $q = 3, 4, 5, 6$, and 7 . The estimation with $q = 6$ provides the best fit.

The effect of burnout on refinancers in a pool is analyzed by observing the path traced by the short-term interest rate. If the short-term interest rate reaches a minimum at any given time since the last previous interest rate adjustment date, some additional mortgagors will be induced to refinance their adjustable rate loans. Because the underlying loans have different adjustment intervals, we use a moving twelve-month minimum of the short-term interest rate to capture burnout.

The variable,

$$r_{\min}(t) = \text{Min}[r(t-12-k), r(t-11-k), \dots, r(t-k)],$$

represents the minimum short-term interest rate over the twelve months prior to month t lagged by k months.

The dummy variable

$$x_b = \begin{cases} 1 & \text{if } r_{\min}(t) < r_{\min}(t-1), \\ 0 & \text{otherwise,} \end{cases}$$

represents the burnout covariate.

The relative change in the spread between the long- and short-term interest rates is used as a proxy for changes in the slope of the yield curve. The change in spread is defined as $x_{sp} = \{[l(t-k) - r(t-k)] - [l(t-k-1) - r(t-k-1)]\}$, where l is the thirty-year Treasury bond rate for month $t-k$, and k is again the number of months the interest rate is lagged.

The effect of interest rate predictions (by switchers) on prepayments is captured by an indicator variable that traces the path of the long-term rate since issuance of the ARM-backed security. If the long-term rate hits a new minimum level since origination of the pool, some additional mortgagors will switch to a fixed rate loan. The variable,

$$l_{\min}(t) = \text{Min}[l(1), l(2), \dots, l(t-k)],$$

represents the minimum long-term rate since the issuance of the ARM-backed security lagged by k months. The indicator variable

$$x_p = \begin{cases} 1 & \text{if } l_{\min}(t) < l_{\min}(t-1), \\ 0 & \text{otherwise,} \end{cases}$$

represents the interest rate prediction variable.

The predictive ability of lifetime caps on prepayments is captured by the difference between the weighted average lifetime cap of the mortgages in a pool and the average lifetime cap across all pools in the sample. That

is, $x_c = [CAP_j - \sum_{j=1}^J CAP_j / J]$ where CAP_j is the weighted average lifetime cap of pool j and J is the number of pools in the sample.

The effect of the interest rate margin on prepayments is measured as the weighted average margin of the mortgages in a pool less the average margin across all pools in the sample. That is,

$$x_m = [WAM_j - \sum_{j=1}^J WAM_j / J]$$

where WAM_j is the weighted average margin of the loans in pool j .

The prepayment model was estimated with interest

rate lags of 0, 1, 2, and 3 months. The model with rates lagged two months provided the best fit. The results of this estimation are presented in column 2 of Table 1. With the exception of the spread between the weighted average lifetime cap of the individual pools and the sample average lifetime cap, the coefficient of each of the variables has the predicted sign, and each is significantly different from zero. The only bothersome result is the sign of the coefficient of the spread between the pool lifetime cap and the sample average cap — according to the regression results, pools with higher lifetime caps tend to prepay more slowly than those with lower caps. This result is contrary to our theoretical argument, and we have no explanation for it.

The coefficients of the other variables do indicate a strong seasonal effect in ARM-backed security prepayments — prepayments speed up in the summer months — and they indicate that mortgagors are sensitive to declines in the short-term rate between coupon rate adjustment dates. They are also consistent with the conjecture that switchers are important in ARM-backed security prepayments. As the spread between the long rate and the short rate increases, prepayments increase. Apparently, as the yield curve steepens, switchers perceive that short-term rates will be higher in the future than previously expected and they are thus motivated to switch to fixed rate mortgages so as to smooth their future cash outflows.

Additionally, when the long-term rate hits a new minimum, prepayments increase. Finally, prepayments also tend to be higher for pools with higher weighted average margins over the index. All these results are consistent with the conjecture that some borrowers select an adjustable rate loan at the outset with the expectation of refinancing into a fixed rate loan at a time in the future when economic circumstances change appropriately.

Because the coefficient of the lifetime cap variable has the wrong sign, the model is reestimated without this variable before estimating the baseline hazards function. These results are presented in column 3 of Table 1. With these coefficients and the monthly prepayment data, the baseline hazards rate of prepayments is estimated for each month since origination of the pools. The baseline hazards rate is then plotted against mortgage age and presented in the Figure. This Figure shows the effect of mortgage age on prepayments in the absence of the effects of the other covariates.

The Figure indicates that the rate of prepayments rises during the first two and a half years, reaching a plateau around the thirtieth month. There is greater dispersion in the baseline prepayment rate in the later months, probably

TABLE 1 • Estimated Coefficients of Proportional Hazards Prepayment • Model for ARM-Backed Securities • (Standard Errors in Parentheses)

(1) Independent Variables	(2) Proportional Hazards Model	(3) Proportional Hazards Model (without lifetime cap)	(4) Age Function	(5) Prepayment Model
Seasonal (x_s)	0.1145 (0.0292)*	0.1116 (0.0296)*		0.1116 (0.0296)*
Δ Short rate (x_r)	-0.3837 (0.1489)*	-0.3674 (0.1512)*		0.3674 (0.1512)*
Burnout (x_b)	0.1388 (0.0368)*	0.1545 (0.0380)*		0.1545 (0.0380)*
Δ Slope of yield curve (x_{sp})	0.0980 (0.0517)**	0.0907 (0.0492)**		0.0907 (0.0492)**
Long rate minimum (x_p)	0.1958 (0.0577)*	0.1850 (0.0603)*		0.1850 (0.0603)*
Pool wtd. ave. lifetime cap - sample ave. lifetime cap (x_c)	-0.0897 (0.0221)*			
Pool wtd. ave. margin - sample ave. margin (x_m)	0.8076 (0.0754)*	0.6731 (0.0701)*		0.6731 (0.0701)*
Age in months			0.0487 (0.0024)*	0.0487 (0.0024)*
(Age in months) ²			-0.0302 (0.0032)*	-0.0302 (0.0032)*
Adjusted R ²			0.97	0.65

* Indicates statistical significance at 0.01 level.

** Indicates statistical significance at 0.05 level.

Variable Definitions

Seasonal (x_s)	=	1 for the months April to August, 0 for the months September to March.
Δ Short rate (x_r)	=	relative change in short-term rate over six months, lagged by two months.
Burnout (x_b)	=	1 if the short-term interest rate reaches a minimum since the beginning of each 12-month interval starting from the origination of the mortgage pool. = 0 otherwise.
Δ Slope of yield curve (x_{sp})	=	one-month change in spread between long- and short-term interest rates lagged by two months.

Variable Definitions

(continued)

Long rate minimum (x_p)	=	1 if the long-term interest rate reaches a minimum since the origination of the mortgage pool.
	=	0 otherwise.
Pool wtd. ave. lifetime cap - sample ave. cap (x_m)	=	difference between the weighted average lifetime cap of the pool and the sample average lifetime cap.
Pool wtd. ave. margin - sample ave. margin (x_m)	=	difference between the weighted average margin of the pool and the sample average margin.
Age in months	=	weighted average number of months since origination of the mortgages in the pool divided by 60 months if the number of months is less than 60.
	=	1 otherwise.

because of the smaller number of pools in the sample for which prepayment data are available over those months.

As the final step in estimation of the prepayment model, a continuous function of time is fitted through the data for the baseline prepayment rate to capture the effect of aging on prepayments. The maximum weighted average age of the pools in our data is fifty-seven months. Thus, we use the transformed variable:

$$\tau = \begin{cases} T / 60 & \text{for } T \leq 60 \text{ months} \\ 1 & \text{for } T > 60 \text{ months} \end{cases}$$

To capture the curvilinear effect of aging, we use the quadratic form $\pi_0(\tau) = c_0\tau + c_1\tau^2$. No intercept term is

employed in this function because, at origination of the mortgages, the prepayment rate is assumed to be zero. The parameter estimates for the mortgage age function are given in the fourth column of Table 1. The estimated coefficients are both significant at the 0.01 level, and the adjusted R^2 of the model is 0.97.

The full model is given in column 5. The full model has an adjusted R^2 of 0.65. Because the data encompass a maximum of fifty-seven months, the effect of age on prepayments is assumed to remain constant after the sixtieth month following origination of a pool. This function is used in the ARM-backed security valuation model.

ARM-BACKED SECURITY VALUATION

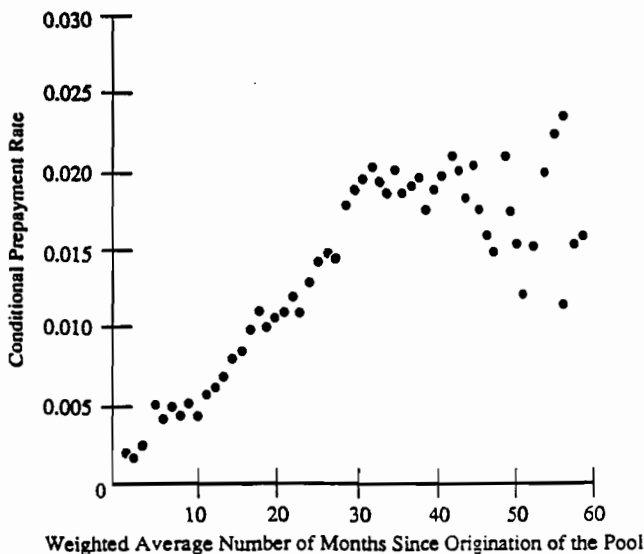
Our model for valuing ARM-backed securities is based on the Brennan and Schwartz [1979] two-factor model of the term structure of interest rates. This model assumes that the term structure can be specified completely by the short-term interest rate, r , and the long-term consol rate, l , with dynamics

$$\begin{aligned} dr &= [a_1 + b_1(l - r)]dt + \sigma_1 r dz_1, \\ dl &= (a_2 + b_2 l + c_2 r)dt + \sigma_2 l dz_2, \end{aligned} \tag{2}$$

where dz_1 and dz_2 are standard Weiner processes with instantaneous correlation coefficient ρ .

The value of an ARM-backed security depends upon the interest and principal payments on the underlying loans. These payments depend in turn upon the stochastic path of the index rate to which the adjustable rate loans are linked and upon the remaining principal balance of the underlying collateral at each time. Because we use the short- and long-term interest rates to specify the yield

FIGURE ■ Plot of Conditional Prepayment Rate for Baseline Hazards Function Against the Weighted Average Number of Months Since Organization of the Pool



curve, the index, $I_0(t)$, to which the adjustable rate loans are linked must be specified as a function of the short- and long-term rate.

For simplicity, we employ a linear function of r and l , i.e., $I_0(t) = a_0r(t) + b_0l(t)$ where a_0 and b_0 are weights applied to the short- and long-term rates. For empirical purposes, a better approximation of the index can be obtained by including historical values of r and l (Ramasmwamy and Sundaresan [1986]). The index is then defined as a weighted average of the historical values of $I_0(t)$ as

$$I(t) = \int_{-\infty}^0 e^{-\gamma s} I_0(t-s) ds \quad \gamma > 0$$

with dynamics

$$dI = \gamma[I_0 - I]dt. \quad (3)$$

The parameter, γ , assigns weights to current and historical values of I_0 in determining $I(t)$.

The interest and principal payments also depend on the fraction of the original mortgages remaining in the pool. Let $y(t)$ represent the fraction of the pool of mortgages surviving until time t in the absence of prepayments with dynamics

$$dy = -\gamma\pi dt, \quad (4)$$

where π denotes the conditional prepayment rate.

Finally, the variable, $x(t)$, which is used to capture the history of the long-term rate, l , is defined as

$$x(t) = \int_{-\infty}^0 e^{-\alpha s} l(t-s) ds \quad \alpha > 0,$$

with dynamics

$$dx(t) = a[l - x]dt. \quad (5)$$

With the five state variables, r, l, I, y , and x , standard arbitrage arguments can be used to generate the differential equation that governs the value, $M(r, l, I, y, x, t)$, of the ARM-backed security:

$$\begin{aligned} & 1/2r^2\sigma_r^2M_{rr} + r\rho\sigma_r\sigma_I M_{rI} + 1/2I^2\sigma_I^2M_{II} + \\ & (a_1 + b_1(l-r) - \lambda\sigma_r r)M_r + I(\sigma_I^2 + l-r)M_I + \\ & \gamma(a_0r + b_0l - I)M_y + \alpha(l-x)M_x - \gamma\pi M_t + \\ & M_t - rM + CF(t) = 0 \end{aligned} \quad (6)$$

where the subscripts on M represent partial derivatives, λ is the market price of short-term interest rate risk, and $CF(t)$ is the continuous cash flow rate to the securityholders at time t . With the assumption that the underlying mortgages are fully amortizing, the terminal condition at maturity, T , for the solution of (6), is $M(r, l, I, y, x, T) = 0$.

The cash flow term, $CF(t)$, includes the scheduled interest and principal payments and any unscheduled princi-

pal prepayments to the securityholders at time t . In a continuous time model with instantaneous adjustment of the interest rate and in the absence of interest rate caps and floors, the coupon payment, $A(t)$, and the remaining principal balance, $F(t)$, at any time t can be expressed as

$$A(t) = I(t)F(t)/[1 - \exp\{-I(t)(T-t)\}], \quad (7)$$

$$F(t) = F(0)\exp\left\{-\int_0^t \frac{I(s)\exp\{-I(s)(T-s)\} ds}{1 - \exp\{-I(s)(T-s)\}}\right\}. \quad (8)$$

The cash flow term can then be expressed as

$$CF(t) = \gamma(t)[A(t) + \pi(t)F(t)]. \quad (9)$$

The cash flow equation (9), the equations determining the remaining principal balance (7 and 8), and the differential equation for the ARM-backed security value (6) comprise a set of equations that can be solved simultaneously to determine the value of the ARM-backed security at any time. In actuality, as we have noted, the interest rate of the security is adjusted at discrete intervals and is subject to both lifetime and annual adjustment caps. To account for these contractual features, the cash flow term must be modified.

For the valuation of an ARM-backed security, it is necessary to specify the particular contractual features of the ARM-backed security as well as those of the underlying collateral. The contractual features of the former are required for determining the scheduled principal and interest payments, while the contractual features of the underlying collateral are required to determine principal prepayments.

Each ARM-backed security has a predetermined initial coupon rate and a set of rules that govern the coupon rate on each adjustment date. The coupon rate on an adjustment date is a function of the rate in the previous period, the current value of the index rate, the margin over the index, and the annual adjustment and lifetime caps. The scheduled principal and interest payment until the next coupon adjustment date are determined according to the new contract rate of interest. The payment is equal to an amount that, if held constant for the remaining term to maturity of the ARM-backed security, would completely pay off the principal at maturity.

Let $t(i, j)$ denote the j^{th} monthly payment date after the i^{th} annual interest rate adjustment date; let $a(i)$ be the coupon interest rate determined on the i^{th} adjustment date; let $I(i, j)$ be the index rate at $t(i, j)$; let m be the margin over

the index that determines the contract interest rate a ; let c_p be the periodic adjustment cap, and let c_l be the lifetime cap. Then, on each interest rate adjustment date, $t(i+1,0)$, the new contract interest rate, $a(i+1)$, on the ARM-backed security is

$$(7) \quad a(i+1) = \text{Max}[\text{Min}\{I(i+1,0) + m, a(i) + c_p, c_l\}, a(i) - c_p]. \quad (10)$$

(8) The sum of the index rate, $I(i+1, 0)$, and the margin, m , would be the new contract rate on an uncapped security. The periodic and lifetime caps are indicated by $a(i) + c_p$ and c_l , and $a(i) - c_p$ is the periodic floor.

(9) Let $F(i,j)$ denote the unpaid principal balance at $t(i,j)$, let $n(i)$ be the monthly contract interest rate, and let T be the maturity of the loan in months (i.e., for a thirty-year security, $T = 360$). Then the scheduled monthly payment (principal plus interest) between any two coupon adjustment dates is

$$(9) \quad A(i) = \frac{n(i)F(i,0)[1 + n(i)]^{T-t(i,0)}}{[1 + n(i)]^{T-t(i,0)} - 1}, \quad (11)$$

where $n(i) = a(i)/12$ and

$$(6) \quad F(i, j) = F(i, 0) - \sum_{s=1}^j [A(i) - n(i)F(i, s-1)]. \quad (12)$$

The values of $F(i,j)$ are determined sequentially, beginning with $F(i,0)$. It is also necessary to transform the equation governing the index into a discrete form as

$$I(t) = \frac{\gamma a_0}{1 + \gamma} r(t) - \frac{\gamma b_0}{1 + \gamma} I(t) + \frac{1}{1 + \gamma} I(t-1),$$

or, equivalently,

$$(13) \quad I(t) = \psi_1 r(t) + \psi_2 I(t) + \psi_3 I(t-1).$$

If ψ_3 equals zero, the index adjusts perfectly to the market so that there is no lagged effect. With the given relationships, the cash flow term in the differential equation governing the ARM-backed security value is

$$(14) \quad CF(i,j) = \gamma [t(i,j)] [A(i) + \pi \{t(i,j)\} F(i,j)].$$

VALUATION RESULTS

Monte Carlo Simulation

Because the second-order partial differential equation for the value of the ARM-backed security cannot be

solved with conventional finite difference methods, Monte Carlo simulation is used (Boyle [1977]). According to lemma 4 of Cox, Ingersoll, and Ross [1985], the equilibrium value of a claim is given by its expected discounted value, with discounting done at the risk-free rate, when the expectation is taken with respect to a risk-adjusted process for all the state variables in the differential equation governing the value of the contingent claim.

The risk-adjusted interest rate paths are generated as follows:

$$dr = (a_1 + b_1(l-r) - \lambda r \sigma_1) dt + r \sigma_1 dz_1, \\ dl = l(\sigma_2^2 + l-r) dt + l \sigma_2 dz_2.$$

The values of r along the interest rate paths generated by Equation (15) are used for discounting cash flows.

Parameter Estimation

To implement the Monte Carlo simulation, it is necessary to specify the parameters of the interest rate process. Monthly data on three-month Treasury bills and thirty-year Treasury bonds for the period 1981 through 1988 are used to estimate the parameters of the discretized form of Equation (2) as in Schwartz and Torous [1989]. Because dz_1 and dz_2 are correlated, we use Zellner's seemingly unrelated regression procedure. The results are shown in panel A of Table 2. With these estimated parameters, Equation (15) is used to simulate the paths of risk-adjusted interest rates.

The parameters for the dynamics of the index rate must also be estimated. The parameters of Equation (13), the discretized form of the dynamics of the index rate, are estimated with monthly interest rate data for the 1981 through 1988 period. As the index employed in the Freddie Mac Series 35 program is the one-year Treasury rate, the one-year rate is regressed against contemporaneous observations of the three-month T-bill rate and the thirty-year T-bond rate and lagged observations of the one-year rate. The estimated coefficients are reported in panel B of Table 2.

Finally, it is necessary to estimate the market price of risk, λ . We use a procedure similar to that used by Schwartz and Torous [1989]. Specifically, we choose for our estimate of λ the value of λ that prices at par an uncapped thirty-year non-callable ARM-backed security that has a coupon rate equal to the one-year Treasury rate plus 1.8%,⁴ an annual interest rate adjustment interval, and for which the amortization schedule is determined for the next twelve months at each coupon adjustment date. Additionally, the values of the one-year Treasury rate, the short-term interest rate, and

TABLE 2 ■ Parameter Estimates of the Two-Factor Model of the Term Structure and the One-Year Treasury Index Model ■ (Standard Errors in Parentheses)

A: Parameter estimates of the two-factor model of interest rates.*

a_1	b_1	a_2	b_2	c_2	σ_1	σ_2	ρ
-0.0019	0.0791	0.0236	0.4386	-0.6098	0.0627	0.0372	0.5771
(0.0011)	(0.0469)	(0.0204)	(0.2317)	(0.3324)			

B: Parameter estimates for the dynamics of the one-year Treasury index.**

Ψ_1	Ψ_2	Ψ_3
0.5465	0.2041	0.2567
(0.0353)	(0.0264)	(0.0447)

*Zellner's seemingly unrelated regression procedure is used to estimate the parameters of the interest rate process $dr = (a_1 + b_1(l-r))dt + \sigma_1 r dz_1$ and $dl = (a_2 + b_2 l + c_2 r)dt + \sigma_2 l dz_2$ with $dz_1 dz_2 = \rho dt$. The coefficients a_1, \dots, c_2 have the dimension of inverse of time (month⁻¹). The standard deviations, σ_1 and σ_2 , have the dimension of inverse of the square root of time (month^{-1/2}), and the correlation coefficient is dimensionless.

**Ordinary least squares regression is used to estimate $I_t = \Psi_1 r_t + \Psi_2 l_t + \Psi_3 l_{t-1} + \epsilon_t$.

the long-term interest rate are assumed to be 10%. The value of λ obtained using this procedure is -0.45.

Simulation Results

Our simulations are designed to illustrate the effect of prepayments on the value of ARM-backed securities with different contractual features and in different economic environments. To accomplish this, we compare ARM-backed security values assuming no prepayments with values generated using our empirically estimated prepayment function.

Because of servicing and guarantee fees, the lifetime caps and margins of ARM-backed securities are less than those of the underlying mortgage loans. In our simulations, the security has a weighted average lifetime cap that is 1.5% less than the weighted average lifetime cap of the underlying loans, and the margin of the security is 1.0% less than the weighted average margin of the underlying loans. These are approximately the average values observed in the Freddie Mac data. The annual adjustment cap is $\pm 2\%$ for both the underlying mortgages and the security. We assume further that the teaser rate is in effect for six months and, without loss of generality, that all loans have 360 months remaining until maturity.

The simulation results are presented in Table 3. Panel A illustrates the effect of prepayments on the value of ARM-backed securities with lifetime caps of 12.5% and 14.5% under different interest rate environments. The short- and the long-term interest rates are varied from 8.0% to 12.0%. These scenarios encompass downward-sloping,

upward-sloping, and flat term structures. Columns 3 and 5 present the values with zero prepayments, and columns 4 and 6 present the values with the empirically estimated prepayment function.

First, the results illustrate the not surprising conclusion that securities with lower caps have lower values and are more sensitive to changes in interest rates than are securities with higher caps. They also demonstrate that, unlike a freely floating security with instantaneous interest rate adjustments whose value should be unaffected by changes in interest rates, a security with interest rate caps and adjustment lags exhibits substantial sensitivity to changes in interest rates.

More important for our purposes, the simulations demonstrate the degree to which prepayments dampen the fluctuations in ARM-backed security values as interest rates change. For example, in the middle section of panel A, when there are no prepayments, the security's value declines from 108.13 to 90.25 as the long-term interest rate increases from 8.0% to 12.0% (for a security with a 12.5% lifetime cap). This range in values is cut in half when prepayments are considered. That is, with the empirically estimated prepayment function, the security's value fluctuates from 103.21 to 94.69 as the long-term rate increases from 8.0% to 12.0%. When prepayments are considered, the interest rate sensitivity of ARM-backed securities much more closely approximates that of "true" adjustable rate securities.

Panel B illustrates much the same point. Panel B shows the effect of different interest rate margins on ARM-backed security values under two different interest rate sce-

TABLE 3 ■ Values of ARM-Backed Securities*

A. Sensitivity to Changes in Interest Rates

(Coupon rate at origination (i.e., teaser rate) = 9.0%; Margin over the index = 1.75%)

Short-Term Interest Rate	Long-Term Interest Rate	Lifetime Cap = 12.5%		Lifetime Cap = 14.5%	
		Zero Prepayments	Empirical Prepayment Function	Zero Prepayments	Empirical Prepayment Function
8.0%	8.0%	108.89	104.39	110.63	104.77
8.0	10.0	100.65	101.06	104.20	102.19
8.0	12.0	91.09	96.21	96.11	98.08
10.0%	8.0%	108.13	103.21	109.78	103.60
10.0	10.0	99.74	99.58	103.14	100.70
10.0	12.0	90.25	94.69	95.09	96.50
12.0%	8.0%	106.94	101.67	108.50	102.07
12.0	10.0	98.58	97.91	101.82	99.01
12.0	12.0	89.34	93.13	93.97	94.88

B. Sensitivity to Changes in the Margin, the Lifetime Cap, and the Slope of the Yield Curve

Interest Rate Margin	Lifetime Interest Rate Cap	Upward-Sloping Yield Curve (Short-term rate = 8.5% Long-term rate = 10.0% Teaser rate = 8.0%)		Downward-Sloping Yield Curve (Short-term rate = 10.0% Long-term rate = 8.5% Teaser rate = 8.5%)	
		Zero Prepayments	Empirical Prepayment Function	Zero Prepayments	Empirical Prepayment Function
1.50%	12.5%	98.30	99.06	104.38	101.67
	13.5	100.11	99.57	106.10	101.98
	14.5	101.64	100.00	107.02	102.20
1.75%	12.5%	99.08	99.40	105.62	101.90
	13.5	101.17	99.88	107.64	102.19
	14.5	102.81	100.28	108.65	102.39
2.00%	12.5%	99.80	99.65	106.80	102.01
	13.5	102.18	100.09	109.13	102.27
	14.5	103.94	100.45	110.23	102.46

*Values are for fully amortizing, one-year Treasury indexed ARM-backed securities with 360 months remaining to maturity. The first coupon adjustment occurs after six months with subsequent adjustments at twelve-month intervals. The weighted average life-of-loan cap on the underlying pool of mortgages is 1.5% points above the lifetime cap of the security, and the weighted average margin of the pool is 1.0% greater than the margin of the security. All of the underlying loans and the ARM-backed security have an annual interest rate adjustment cap of $\pm 2.0\%$.

narios — an upward-sloping yield curve and a downward-sloping yield curve. First, as expected, the lower the interest margin, the lower the security's value. Second, regardless of whether the yield curve is upward- or downward-sloping, prepayments push the value of the security toward par value. This effect is much more dramatic when the term

structure is downward-sloping and when the security has a higher lifetime cap.

CONCLUSION

We have estimated a prepayment model for adjustable rate mortgage-backed securities to develop a valuation model. Adjustable rate mortgage prepayers are categorized as relocators (individuals who pay off their loans to move to another location), refinancers (individuals who pay off their loans to take out another adjustable rate loan at a lower index rate), or switchers (individuals who pay off their adjustable rate loans to switch to a fixed rate mortgage). The model is estimated with prepayment data from the Freddie Mac Series 35 ARM-backed securities program.

Results are consistent with the conjecture that relocation, refinancing, and switching are all important motives for ARM-backed security prepayments. Specifically, ARM-backed security prepayments depend upon the age of the mortgage, the season of the year, changes in the short-term interest rate, changes in the slope of the yield curve, the history of interest rates, and the weighted average margin of loans in the pool.

ENDNOTES

¹Previous work most closely related to ours is Ramaswamy and Sandarsen [1986], Kau, Keenan, Muller and Epperson [1990], and Berk and Roll [1988].

²This choice may also involve a trade-off between the margin and the annual adjustment or lifetime cap, but in our sample all the loans have the same annual adjustment (i.e., $\pm 2\%$) cap.

APPENDIX

Estimation of the Proportional Hazards Model

The method of partial likelihood was developed by Cox [1975] and can be used to estimate the parameters of a proportional hazards model in a two-step procedure. In the first step, the effect of time on prepayments is eliminated. Let $R(tg)$ denote the set of mortgages outstanding at any time, tg , in a pool. The conditional probability that mortgage g prepays at time tg , given that the set of mortgages $R(tg)$ are exposed to prepayment risk, and that exactly one prepayment occurs at tg , is

$$\frac{\pi(tg; v_g, \theta)}{\sum_{g \in R(tg)} \pi(tg; v_g, \theta)} = \frac{\exp(xg\beta)}{\sum_{g \in R(tg)} \exp(xg\beta)} \quad A(1)$$

for $g = 1, \dots, G$.

The partial likelihood function of β is found by taking the product over all prepayment points, tg , to give

$$L(\beta) = \prod_{g=1}^G \left(\frac{\exp(xg\beta)}{\sum_{g \in R(tg)} \exp(xg\beta)} \right) \quad A(2)$$

Two important conclusions emerge. The first is that prepayments can have a significant effect on the value of ARM-backed securities, and failure to recognize this effect can lead to improper pricing decisions by market participants. Second, prepayments substantially dampen the price sensitivity of ARM-backed securities to changes in interest rates.

This second conclusion has important implications because, on the one hand, adjustable rate loans have been proposed as a mechanism for reducing the interest rate sensitivity of a financial institution's asset portfolio. On the other hand, critics have noted, quite correctly, that the values of ARMs that are subject to caps and interest rate adjustment lags are still quite sensitive to changes in interest rates.

Our simulation results support that conclusion when prepayments are not considered. When prepayments are considered, however, the sensitivity of ARM-backed securities to interest rate changes more closely approximates that of "true" adjustable rate securities. In that regard, prepayments are an important consideration in the analysis of the interest rate sensitivity of a financial institution's asset portfolio.

³The model was also estimated under the assumption that each mortgage has a face value of \$50,000 at origination. The model coefficients are insensitive to this change in the assumed face amount of the loans.

⁴We use 1.8% because the sample average margin over the index of the ARM-backed securities in the Freddie Mac data is 1.80%.

Ties in prepayments imply that, for each prepayment time represented by a month, there may be several mortgages prepaying in a pool. If d_g denotes the number of prepayments in a pool at time, t_g , the partial likelihood function is expressed as

$$L(\beta) = \prod_{g=1}^G \left(\frac{\exp(sg\beta)}{\sum_{g \in R(t_g)} \exp(sg\beta)} \right) \quad A(3)$$

In Equation (A3), s_g is the sum of the covariate vectors associated with the d_g prepayments at t_g , $s_g = \sum_{i=1}^{d_g} s_{gi}$. The

set $R_{d_g}(t_g)$ consists of all subsets of d_g mortgages chosen from the set outstanding, $R(t_g)$, without replacement. Because the mortgages prepaying are indistinguishable in the data set, all the d_g mortgages have the same covariates, such that, $s_g = d_g x_g$. This form of the likelihood function poses serious computational difficulties. An

approximation to this function is

$$L(\beta) = \prod_{g=1}^G \frac{\exp(x_g \beta)}{[\sum_{r \in R(g)} \exp(x_r \beta)] dg} \quad A(4)$$

Kalbfleisch and Prentice [1980] observe that if the number of ties in each period is small compared to the number outstanding, which is true in our case, the partial likelihood function can be well approximated.

We use new notation for the discretized form of the likelihood function. Let d_{ij} be the number of mortgages prepaying in the j^{th} pool at the i^{th} month from origination of the mortgages, and let n_{ij} be the number of mortgages outstanding at the beginning of each month. For each observation period i , there are J_i pools of mortgages in the sample, and i ranges from 1 to I . The approximate value of the log-likelihood function is

$$\log L = \sum_{i=1}^I \left[\sum_{j=1}^{J_i} d_{ij}(x_{ij}\beta) - \left(\sum_{j=1}^{J_i} d_{ij} \right) \log \left(\sum_{j=1}^{J_i} n_{ij} \exp(x_{ij}\beta) \right) \right] \quad A(5)$$

The likelihood function is then maximized by equating its first derivatives with respect to the parameters β_k ($k = 1, \dots, K$), to zero:

$$\frac{\partial \log L}{\partial \beta_k} = 0, \quad K = 1, \dots, K. \quad A(6)$$

This gives a system of K simultaneous non-linear equations with an equal number of unknowns. The equations are solved using an iterative Newton-Raphson procedure. Note that the likelihood function may have several local maxima, but because the covariates are the deviations from sample means, the

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- The next step is to estimate the baseline hazards function. The estimates of β s obtained from the partial likelihood estimation procedure are used in this step. Let λ_i be the baseline hazards values for each time period i , $i = 1, \dots, I$. Define $\alpha_i = 1 - \lambda_i$. After determining the estimates of the parameters, β , the overall likelihood function for the surviving distribution function for mortgages can then be written as
- $$\prod_{i=1}^I \prod_{j=1}^{J_i} \left[(1 - \alpha_i)^{\exp(x_{ij}\beta)} (\alpha_i)^{\exp(x_{ij}\beta)} n_{ij}^{-d_{ij}} \right] \quad A(7)$$
- Differentiating the log of the likelihood function for the baseline hazards rate, we obtain an equation for each time period, $i=1, \dots, I$.
- $$\sum_{j=1}^{J_i} \frac{d_{ij} \exp(x_{ij}\beta)}{1 - \alpha_i \exp(x_{ij}\beta)} = \sum_{j=1}^{J_i} n_{ij} \exp(x_{ij}\beta) \quad A(8)$$
- This equation is solved for a_i for each month i using an iterative procedure. The values of $1 - a_i$ are the baseline hazards estimates for each time period. A smooth curve is then fitted through these points to obtain a continuous function.
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