Financing newsvendor inventory

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A B S T R A C T
If the cost of borrowing is not too high, the capital-constrained newsvendor borrows funds to procure an amount that is less than would be ideal. The lender charges an interest rate that decreases in the newsvendor’s equity. Furthermore, we derived a non-linear loan schedule that coordinates the channel.

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1. Introduction

When faced with uncertain demand, effective inventory management entails procuring a sufficient amount of stock to buffer against variations in demand from its mean. Typically, the cost of holding inventory, including its financing, is balanced against other costs that include those of order processing and order fulfillment. In the vast literature on this fundamental topic, researchers almost inevitably assume that while inventory is costly, there is sufficient working capital available for its financing. Such a prototypical case arises when working capital is provided by corporate headquarters of large firms to finance divisional inventory at a pre-specified hurdle rate.

While a successful large corporation may be in a position to adequately fund inventory, there are many situations in which decision makers face liquidity constraints. An important case arises when a firm is nascent and is therefore, necessarily, liquidity-constrained. The recent documented difficulties of PenAgain, one such start-up, are detailed in a story by [4] in the Wall Street Journal. For complementary reasons, a firm under financial distress or bankruptcy protection also finds itself capital-constrained. In such cases the firm resorts to factors or debt-in-place financing to fund its illiquid operations. And, in many international transactions, otherwise well-financed entities find it convenient to purchase goods by executing letters-of-credit, which bind financial intermediaries to ensure that suppliers honor purchase agreements.

The gist of these situations is that the procurement of inventory is done with a mixture of equity and debt. Some natural questions that arise include: (1) How does the bank determine the interest rate? (2) How much should be borrowed? (3) How is this transaction influenced by the equity available? And, (4) how are the terms of the transaction influenced by the nature of demand uncertainty? The goal of this paper is to study the structure of such a transaction by focusing on the case of a one-time purchase of a seasonal good.

We use the setting of the inventory procurement problem of a capital-constrained newsvendor (CCNV). Given a set purchase and selling price, and knowing the distribution of seasonal demand, the CCNV is unable to order that quantity which satisfies the newsvendor fractile. Therefore, it must decide how much to borrow at a given interest rate to finance additional procurement. Since demand is uncertain, it is possible that sales will not be high enough to pay off the loan at the end of the selling season. Thus, the bank must take into account the possibility of default when determining the interest rate to charge.

We represent the strategic interaction between the two parties as a Stackelberg game in which the bank is the leader and the CCNV is the follower. This setting captures instances when specialized knowledge precludes existence of market efficiency. Our results show that if the cost of borrowing is not too high, the CCNV borrows funds to procure an amount that is less than the channel optimal quantity. In return the bank charges an interest rate that decreases in the equity position of the firm. Our numerical work, which is not reported here, further suggests that efficiency is relatively high and tends to increase with decreasing skewness of the demand distribution, with the mean fixed. However, this loss in efficiency can be eliminated if a profit-splitting non-linear loan mechanism is invoked.

Until recently financial considerations were conspicuously scarce in the extensive literature on models of inventory and production processes. However, research on the coordination of operational and financial decisions has been growing. Start-up
firms with a criterion of maximizing survival probability, according to [1], should be more cautious in their component purchasing strategy than well-established ones, and their purchasing quantity is not necessarily monotone in their available capital. Ref. [2] examines the capacity expansion decisions of a firm so that its expected proceedings from an initial public offering of stock is maximized. Ref. [11] studies the coordination of operations with finance in a dynamic setting. Ref. [3] work on the impact of risk on capacity planning models, [13] incorporate bankruptcy deadweight costs in the context of multi-period inventory models, and [9,8,10] study the value of operational flexibility, and [7] study hedging demand risk in inventory decisions. Our work is also related to the supply chain coordination literature on which [6] is an excellent review.

The most closely related papers are [12,5]. In a non-strategic framework, [12] has also used a single-period newsvendor problem to study the transaction that finances inventory. In contrast to our model, theirs assumes that the cost of capital of the bank and the newsvendor are both fixed at the risk-free rate, which is most appropriate when markets are efficient and complete. Consequently, as discussed immediately after Proposition 2 in Section 3, this optimal order quantity does not depend on the CCNV’s level of equity. Their single-period model is further enriched by including the impact of taxes and bankruptcy costs, features that can be incorporated seamlessly in our model.

In contrast to the case for [12], a key feature of our model, as in [5], is that the interest rate to be charged by the bank is endogenous. Moreover, as in the work of [11], the newsvendor’s problem may be modeled as a multi-period problem that explicitly examines the cost of reorganization when bankruptcy risks are significant. Our single-period model could be used as a building block for considering such models when liquidity or working capital is an issue.

Importantly, our work and [12] may be viewed as single-period instances of the dynamic aspects of asset management financing originally considered by [5]. In particular, our Proposition 1 in Section 2 is a distilled version of their Theorem 3. While some of our analysis has a similar flavor to theirs, the key distinction is that we formally treat the bank as the Stackelberg leader and prove that the Stackelberg equilibrium is unique. In contrast, there are many retailers in [5]. Their bank behaves more like a monopolist who offers take-or-leave-it loans to retailers with different internal capitals.

[14] allows a manufacture to offer loans and provide goods to its capital-constrained retailers in a monopoly setting. They examine whether the typical supply chain contracts could achieve channel coordination. In contrast, our bank provides a loan to the newsvendor but not physical goods. We show that a non-linear loan schedule can coordinate the channel.

We model the problem faced by the CCNV in Section 2.1 and the bank’s problem in Section 2.2. Some comparative statics of the equilibrium are presented in Section 2.3. In Section 3 we propose a non-linear loan schedule to achieve channel coordination.

2. The Stackelberg game

2.1. The newsvendor’s problem

The newsvendor places an order for Q units at unit cost c before the selling season starts and sells them at unit price p, where (p > c). The seasonal demand D follows a cumulative probability distribution F(·) and density probability distribution f(·) on R+. Define F(x) = 1 − F(x). Conveniently assuming a zero salvage value for any excess inventory and zero goodwill costs for lost sales, the classical newsvendor’s optimal order quantity, denoted by Qopt, satisfies F(Qopt) = c/p. However, since the newsvendor is capital-constrained, i.e., its internal capital (henceforth addressed as equity to be differentiated from debt), represented by η, is less than cQopt, the CCNV may find it profitable to finance additional procurement by borrowing B from a financial intermediary, such as a bank, at an interest rate, r. The CCNV borrows just enough to purchase the desired quantity Q at the financing rate r; thereby B = cQ − η. To secure the loan, the CCNV pledges the initial (1+r)B of its revenue to the bank and receives the residual revenue, if any, after sales revenue realizes. Thus the CCNV’s optimization problem is

\[
\max \pi_{\eta} = -\eta - B(1+r)\bar{F}(B/(1+r)) + p\int_{B/(1+r)}^{Q} xdf(x) + pQ\bar{F}(Q).
\]

It provides insight and is practical to make the following changes of variables:

\[
m = \frac{p}{1+r} \quad y = \frac{(cQ - \eta)(1+r)}{p} = \frac{B}{m}.
\]

Note that m represents the discounted revenue from one unit of sales and that y is the number of units that must be sold to fully pay off the loan B taken by the newsvendor. Hence, the CCNV defaults with probability \(F(y)\).

Using (1), the CCNV’s objective function becomes

\[
\pi_{\eta} = -\eta + p\int_{0}^{Q} (x - y)dF(x) + p\int_{y}^{\infty} dF(x)
\]

\[
= -\eta + p\int_{0}^{Q} \bar{F}(x)dx.
\]

In contrast to the formulation for the unconstrained newsvendor, the lower limit, y, of the integral captures the revenue which is in excess of the amount pledged to the bank and the first term recognizes the procurement which is financed by equity. Our formulation of the newsvendor’s problem is a variation of the retailer’s problem considered by [5]. They (Theorem 3, p. 1283) have shown that if the demand distribution has increasing failure rate (IFR), the optimal solution can be fully characterized by manipulating the Karush–Kuhn–Tucker conditions to yield:

**Proposition 1.** For any \(\eta < cQ_{\theta}\), if the demand distribution function, \(F(·)\), is IFR (has an increasing failure rate), then the newsvendor’s optimal ordering quantity, \(Q^*\), is determined as follows:

\[
Q^* = \begin{cases} \frac{\eta}{c} \cdot \bar{F}^{-1}\left(\frac{c}{m}\right) & \text{if } \eta/c > \bar{F}^{-1}\left(\frac{c}{m}\right) \\ Q, & \text{otherwise} \end{cases}
\]

where

\[
m\bar{F}(\hat{Q}) = c\bar{F}(y).
\]

In the first case of (3), the CCNV exhausts its own equity \(\eta\) but does not borrow from the bank. This occurs when the interest rate is too high, suggesting that for every \(\eta\) there is an upper limit on the rate that the bank may charge. It also readily follows that this upper limit is decreasing in the CCNV’s equity. Alternatively, if the interest rate is sufficiently low, after exhausting its equity, the CCNV would seek additional financing to order \(\hat{Q}\), which uniquely satisfies (4).

Given the CCNV’s best response to the interest rate set by the bank, we will next consider the bank’s problem. As the Stackelberg leader, it must take into account how the CCNV will respond to a
change in the interest rate \( r \) or \( m \). This response, from applying the Implicit Function Theorem to (4), is given by

\[
dy = \frac{c\hat{F}(\eta + my)/c - myf(\eta + my)/c}{m^2f(\eta + my)/c - c^2f(y)}.
\]

From the equation above, it can be determined that neither \( y \) nor the default probability is monotone in \( y \), but we show in the next section that at the equilibrium point \( y \) is monotone in \( m \).

2.2. The bank’s problem

At the second stage of the Stackelberg game, the CCNV decides how much to borrow given the interest rate \( r \) charged by the bank. Thus, at the first stage, anticipating the newsvendor’s response, the bank determines \( r \) or \( m \) to maximize its expected profit. This yields the counterpart to (2)

\[
\max_r \pi_b = B (1 + r) \hat{F}(y) + p \int_0^\gamma x dF(x) - B.
\]

The first term of (6) is the interest plus the principal that the bank receives when the CCNV does not default, while the second term is the expected sales revenue when the newsvendor defaults. It follows from Proposition 1 that the CCNV does not borrow if \( m < \frac{c}{\hat{F}(\eta)/c} \). In order to induce the CCNV to borrow, the bank chooses \( m \) between \( \frac{c}{\hat{F}(\eta)/c} \leq m < p \) where \( m < p \) is equivalent to \( r > 0 \). Substituting (1), (6) can be rewritten as

\[
\max_{\frac{c}{\hat{F}(\eta)/c} \leq m < p} \pi_b = \hat{F}(y) \int_0^\gamma x dF(x) - my.
\]

To understand the bank’s problem, one way to proceed is to take the partial derivative of \( \pi_b \) with respect to \( y \) and compute the first-order condition of \( \pi_b \) with respect to \( m \) which yields the first-order condition:

\[
\frac{dy}{dm} = -y' (m) + \left[ -m + \hat{p} (y'(m)) \right] \frac{dy}{dm} = 0
\]

where \( \frac{dy}{dm} \) is given by (5). Now recall from (4) of Proposition 1 that in the admissible range of \( \hat{Q} \leq Q_0 \). Hence, \( c/p = \hat{F}(Q_0) < \hat{F}(\hat{Q}) = c/mF(y'(m)) \). This allows us to conclude that the bank will select \( m \) such that \( m - \hat{p} (y') < 0 \). As a consequence, we can conclude immediately that at equilibrium \( \frac{dy}{dm} \) is strictly monotone. These conclusions are summarized as follows.

Proposition 2. If \( F(\cdot) \) is IFR, (a) the Stackelberg game played out between the bank and the newsvendor has a unique equilibrium point \((y^*, m^*)\) which satisfies (3), (5) and (8), and (b) \( 0 < y^* < \hat{F}^{-1}(m^*/p), \hat{F}^{-1}(c/m^*) < \hat{Q} < Q_0 \) and \( 0 < \frac{dy}{dm} < \frac{dy}{dm^*} \).

Part (b) of Proposition 2 is in contrast to the approach of [12], who assume that both the bank and the CCNV expect a return at the risk-free rate \( r \), resulting in \( \hat{Q} = Q_0 = \hat{F}^{-1}(c/p(1 + r)) \). Here the term \( 1 + rt \) reflects an adjustment to \( p \) for the time value of money, so it suggests a decoupling of procurement and financing decisions. In contrast, we have just demonstrated that the procurement decision is directly linked with financial decisions. Moreover, at equilibrium the CCNV purchases less than would a traditional newsvendor.

2.3. Some comparative statics

Having established that the equilibrium is unique, we are in a position to perform comparative statics. This yields the following comparative statics with respect to \( \eta, p \) and \( c \).

Proposition 3. (a) As \( \eta \) increases, \( y^* \), \( B^* \), and \( r^* \) decrease, \( m^* \) increases. (b) For a given \( \eta \), as \( p \) increases, \( y^* \), \( B^* \), \( r^* \), and \( m^* \) increase. And (c) for a given \( \eta \), as \( c \) increases, \( y^* \) increases, but \( \hat{Q} \), \( m^* \) and \( B^* \) decrease.

Proof. Applying the Implicit Function Theorem to (4) yields

\[
\frac{dm}{d\eta} = \frac{mf(\hat{Q}) - my^*f(\hat{Q})}{cF(\hat{Q}) - my^*f(\hat{Q})}.
\]

Since

\[
\frac{dy}{dm^*} = \frac{c\hat{F}(\eta) - m^*y^*f(\hat{Q})}{f(y^*) - m^*f(\hat{Q})} > 0
\]

at equilibrium, \( c\hat{F}(\eta) - m^*y^*f(\hat{Q}) > 0 \). Therefore, \( \frac{dm^*}{d\eta} > 0 \). Consequently, \( \frac{dy^*}{d\eta} < 0 \) and \( \frac{dB^*}{d\eta} < 0 \). Similarly, applying the Implicit Function Theorem to (8) yields

\[
\frac{dy^*}{dp} = \frac{\hat{F}(y^*)}{m^*} \frac{dy^*}{dm^*} > 0 \quad \frac{dm^*}{dp} = \frac{\hat{F}(y^*)}{m^*} > 0.
\]

For part (c), because \( \frac{dy^*}{dm^*} > 0 \), \( B^* = m^*y^* \), and \( c\hat{Q} = \eta + B^* \), the monotone property of \( Q \) with respect to \( p \) is immediate. Substituting \( p/(1 + r^*) \) for \( m^* \) in (8) and applying the Implicit Function Theorem yields

\[
\left[ -\frac{1}{1 + r^*} + \hat{F}(y^*) \right] \frac{dy^*}{dm^*} dp = \frac{p}{(1 + r^*)^2} \frac{dy^*}{dm^*} dr = 0.
\]

Hence,

\[
\frac{dr^*}{dp} = \frac{\hat{F}(y^*) - \frac{1 + r^*}{1 + r^*}}{m^* + p\hat{F}(y^*)} > 0
\]

since \( \hat{F}(y^*) > 1/(1 + r^*) \). Another way to prove \( \frac{dr^*}{dp} > 0 \) is to observe that since \( \frac{dm^*}{dp} = \frac{\hat{F}(y^*)}{1 + r^*} < 1 \) and \( r = p/m^* + 1 \),

\[
\frac{dr^*}{dp} = \frac{m^* - p\hat{F}(y^*)}{m^* > 0}
\]

because \( p\hat{F}(y^*) > m^* \). The monotone properties of \( y^* \), \( \hat{Q} \), \( B^* \), and \( m^* \) with respect to \( c \) can be proved in the same fashion.

If the CCNV has more equity, it needs to borrow less. Hence, to induce the newsvendor to borrow, the bank has to lower its interest rate or, equivalently, raise \( m \). Therefore, \( B^* \) and \( y^* \) are lower if \( \eta \) is greater. However, surprisingly, the order quantity \( Q \) is not necessarily increasing in \( \eta \). To examine this, applying the Implicit Function Theorem to (4) yields

\[
\frac{d\hat{Q}}{d\eta} = \frac{\hat{F}(\hat{Q}) \frac{dm^*}{d\eta} + f(y^*) \frac{dy^*}{d\eta}}{m^* f(\hat{Q})}.
\]

Because \( \frac{dm^*}{d\eta} > 0 \) and \( \frac{dy^*}{d\eta} < 0 \), the sign of \( \frac{d\hat{Q}}{d\eta} \) is not invariant, as can be confirmed by numerical examples.

For a fixed \( \eta \), a higher \( p \) implies that the compensation that the bank receives is greater when the newsvendor defaults, while its revenue is fixed at \( B(1 + r) \) otherwise. Hence, the bank is willing to offer a lower \( r \) (higher \( m \)). Since \( y^* \) is increasing in \( m \) at equilibrium, \( y^* \) must be also increasing in \( p \). So is \( B^* \) since \( B^* = m^*y^* \). The interest rate that the bank charges, however, is increasing in \( p \) because the absolute value of the marginal decreasing rate of \( m \).
with respect to \( p \) is less than one and \( r = p/m - 1 \). Not surprisingly, both the newsvendor and the bank benefit from a higher unit selling price of the product since the expected profits for both are increasing in \( p \).

For a fixed \( \eta \), a higher \( c \) yields a lower profit margin, thereby a higher threshold above which the CCNV survives, i.e., a greater \( \gamma^* \). Consequently, the bank charges a higher \( r \) (lower \( m \)), and the CCNV borrows and purchases less.

Although the uniqueness of equilibrium is easily established, explicit expressions for \((\gamma^*, m^*)\) are not easily available. We conducted a variety of numerical studies, not reported here, showing that generally as we would expect as the demand becomes less certain (as captured by the coefficient of variation), overall performance improves. In particular, our results show that channel efficiency, the percentage of expected profit relative to the coordinated solution, exhibits multi-modal patterns. This leads us to examine how coordination could be achieved by changing the specification of the loan schedule.

### 3. Coordinating loan schedules

Thus far we have assumed that if the CCNV does not obtain additional financing, it can only order \( Q_0 = \eta/c \) units yielding an expected profit of \( \pi^0_n \). If the CCNV borrows at equilibrium, it borrows sufficient funds to procure an amount somewhat more than \( \eta/c \) but less than the channel optimal quantity \( Q_0 \) which yields the optimal expected channel profit \( \pi^0 \). Thus, at equilibrium there is some loss of channel efficiency. This loss of efficiency arises because we have assumed that the CCNV responds to the rate \( r \) or \( m \) set by the bank by choosing a loan amount \( B \) or \( y \). In this section, we assume that the bank proposes an appropriate non-linear loan schedule \( r(B) \) or equivalently \( m(y) \) to induce the CCNV to order the channel optimal quantity \( Q_0 \). Since now \( m \) is a function of \( y \), if the CCNV borrows, then the first-order condition of \( \pi_n \) with respect to \( y \) yields

\[
F(Q) \left( m + y \frac{dm}{dy} \right) = cF(y).
\]

For this schedule to achieve coordination it must be that \( F(Q) = c/p \) so that

\[
m + y \frac{dm}{dy} = \frac{dB}{dy} = pF(y).
\]

Define

\[
g(y) = \int_0^y F(x)dx.
\]

Then (9) becomes

\[
B = cQ_0 - \eta = my = pg(y) - K
\]

where the range and the meaning of \( K \) will be specified later. Hence, at equilibrium, \( \pi^*_n \), the CCNV’s optimal expected profit is

\[
\pi^*_n = -\eta + p \int_0^Q F(x)dx
\]

where by construction \( \pi_0 = pg(Q_0) - \eta - my \). And \( \pi^*_n \), the bank’s expected profit, is

\[
\pi^*_b = p \int_0^y F(x)dx - my = pg(y) - my = K.
\]

Note that \( my \) is the loan, i.e., the bank’s cost, and \( pg(y) \) is the expected cash flow that the bank receives when the CCNV defaults. Adding (11) and (12) yields the expected total channel profit:

\[
\pi^*_n + \pi^*_b = p \int_0^Q F(x)dx - cQ_0 = pg(Q_0) - Q_0 = \pi_0
\]

where \( cQ_0 = \eta + my \). Define \( \pi^*_n \) as the CCNV’s expected profit without borrowing:

\[
\pi^*_n = -\eta + p \int_0^y F(x)dx = -\eta + pg(\eta/c) < \pi_0.
\]

It is clear now that \( K \) is the share of expected profit that the bank enjoys from the transaction. Furthermore, since \( \pi^*_n \) is increasing in \( \eta \) for \( 0 < \eta < cQ_0 \) and \( \pi_0 \) is fixed, for any transaction between the two parties to occur and for the bank to have incentives to achieve coordination, \( 0 < K < \pi_0 - \pi^*_n \) since \( \pi^*_n \) is increasing in \( \eta \), the upper bound on \( K \) is decreasing in \( \eta \). If \( K = 0 \), then \( \pi^*_n = pg(\eta) - my = 0 \). Hence, like for the well-known channel coordinating two-part tariff, we could interpret \( m \) as the price at which the bank offers the CCNV the loan \( cQ_0 - \eta \) and it breaks even, and \( K \) as the expected fee that it charges the CCNV to extract a profit from lending. The results are summarized as follows.

**Proposition 4.** If the bank offers a loan schedule \( cQ_0 - \eta = p \int_0^y F(x)dx - K \), then \( \pi^*_n = \pi_0 - K \) for \( 0 < K < \pi_0 - \pi^*_n \) and \( 0 < \eta < cQ_0 \), and the channel achieves coordination.

In addition, under the non-linear loan schedule proposed above, it is easy to show that the comparative statics of Proposition 3 continue to hold. However, by design, \( Q^* = Q^0 \), the channel’s first best solution, so it is invariant to \( \eta \). And, the channel efficiency is 100%, so the multi-modal patterns found in our numerical examples cannot arise.

We now illustrate the above coordinating schedule with two examples. Let \( p = 2 \) and \( c = 1 \). First, let demand be uniformly distributed between 0 and 1. So \( Q_0 = 0.5 \), \( \pi_0 = 0.25 \), and \( g(y) = y - y^2/2 \). Hence, 0.5 > \( 2y - y^2 - K \). Solving the quadratic equation yields the loan schedule \( y^* = 1 + \sqrt{0.5 + 2 - K} \). Thus, for any transaction between \( 0 < K < 0.25 - \eta + \eta^2 \) and \( 0 < \eta < 0.5 \). Second, if demand follows the exponential distribution with mean 1, then \( Q_0 = \ln 2 \), \( \pi_0 = 1 - \ln 2 \), \( g(y) = 1 - e^{-y} \), and \( \pi^*_n = 2 - \eta - 2e^{-\eta/2} \). For \( 0 < \eta < 1 \), the loan schedule is \( y^* = -\ln(2(\ln 2 + \eta)) \) where \( 0 < K < \pi_0 - \pi^*_n \) and \( 0 < \eta < 2 \).

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**References**