European Journal of Operational Research 241 (2015) 412–421

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Stochastics and Statistics

Optimum step-stress accelerated degradation test for Wiener degradation process under constraints

Cheng-Hung Hu [a], Ming-Yung Lee [b,*], Jen Tang [c]

[a] Department of Industrial and Information Management, National Cheng Kung University, Tainan City, 701, Taiwan
[b] Statistics and Informatics Department, Providence University, Taichung City, 43301, Taiwan
[c] Krannert Graduate School of Management, Purdue University, 403 West State Street, West Lafayette, IN 47907-2056, United States

ARTICLE INFO

Article history:
Received 28 May 2013
Accepted 8 September 2014
Available online 22 September 2014

Keywords:
Reliability
Accelerated Degradation Test
Step-Stress ADT
Wiener process
Inverse Gaussian distribution

ABSTRACT

To assess a product's reliability for subsequent managerial decisions such as designing an extended warranty policy and developing a maintenance schedule, Accelerated Degradation Test (ADT) has been used to obtain reliability information in a timely manner. In particular, Step-Stress ADT (SSADT) is one of the most commonly used stress loadings for shortening test duration and reducing the required sample size. Although it was demonstrated in many previous studies that the optimum SSADT plan is actually a simple SSADT plan using only two stress levels, most of these results were obtained numerically on a case-by-case basis. In this paper, we formally prove that, under the Wiener degradation model with a drift parameter being a linear function of the (transformed) stress level, a multi-level SSADT plan will degenerate to a simple SSADT plan under many commonly used optimization criteria and some practical constraints. We also show that, under our model assumptions, any SSADT plan with more than two distinct stress levels cannot be optimal. These results are useful for searching for an optimum SSADT plan, since one needs to focus only on simple SSADTs. A numerical example is presented to compare the efficiency of the proposed optimum simple SSADT plans and a SSADT plan proposed by a previous study. In addition, a simulation study is conducted for investigating the efficiency of the proposed SSADT plans when the sample size is small.

1. Introduction

Continued advances in manufacturing technology, coupled with consumer desire for high quality products, have prompted the industry to design and manufacture products that can operate without failure for years. However, for such highly reliable products, it is not an easy task to assess the product reliability within short test durations because sufficient lifetime data are generally required to precisely estimate product's lifetime (failure time, time-to-failure) distribution. Precise reliability estimation is an important part of subsequent managerial decisions such as determining the burn-in time (Sheu & Chien, 2005; Tsai, Tseng, & Balakrishnan, 2011; Ye, Shen, & Xie, 2012), establishing a warranty or maintenance policy (Chien, 2008; Jung & Park, 2003), or pricing extended warranties. To increase the likelihood of observing failures, Accelerated Life Test (ALT) is commonly used by exposing and testing products under a higher stressed condition (e.g., higher temperature, voltage, pressure, vibration, electric current, etc.). Seo, Jung, and Kim (2009) proposed accelerated life test sampling plans satisfying producer's and consumer's risk requirements for deciding the lot acceptability. However, for many highly reliability products, it might still be difficult to obtain enough failure data with short test duration even if an ALT is used. For products like these, an Accelerated Degradation Test (ADT) provides an alternative effective tool to estimate the lifetime. ADT has been successfully applied to many modern products like Light-Emitting Diodes (LED), as in the study by Pan and Crispin (2010).

In an ADT, units are exposed to a relatively severe environment. However, in addition to collecting exact failure time data, measurements of a certain product Quality Characteristic (QC) are recorded at various inspection times. The QC usually degrades (or increases) over time and the lifetime of the product is normally related to the level of the QC. For example, the life of an alloy can be defined when its crack (the QC) reaches size 1.6 inches (Meeker & Escobar, 1998). The life of a certain self-regulating heating cable is related to its resistance (Whitmore & Schenkelberg, 1997). For some elastomers, which are critical materials for hoses and dampers, the life is related to its hardness measure (Elsayed, 2012).

Since the QC of a product degrades over time, the product's life can then be defined as the first-passage time when the QC crosses a

* Corresponding author. Tel.: +886 4 2632 8001x15501.
E-mail addresses: chhu@mail.ncku.edu.tw (C.-H. Hu), mylee@pu.edu.tw (M. Lee), jtang@purdue.edu (J. Tang).

http://dx.doi.org/10.1016/j.ejor.2014.09.003
0377-2217/© 2014 Elsevier B.V. All rights reserved.

pre-specified threshold. ADT data normally consists of measurements of the QC at each measuring time under different stress levels and possible failure times. After ADT data is collected, a statistical model is required for analyzing the observed degradation data and to estimate the product's lifetime under use condition. There are many different models in the literature for fitting a degradation path; for example, the mixed effects nonlinear regression model (Lu & Meeker, 1993; Zhou, Gebraeel, & Serban, 2012), the Gamma process model (Tsai et al., 2011; Tseng, Balakrishnan, & Tsai, 2009), the inverse Gaussian process model (Wang & Xu, 2010), the linear and exponential-based degradation model (Si, Wang, Chen, Hu, & Zhou, 2013), and the Wiener process model (Doksum & Hoyland, 1992; Lee & Tang, 2007). Si, Wang, Hu, and Zhou (2011) provided a review on many stochastic models for estimating the remaining useful life. In this paper, we consider the Wiener process to model the degradation path of a product's QC. Under this assumption, it is well-known that the product's life follows an Inverse Gaussian (IG) distribution. The Wiener/IG model has been used for many applications in a variety of studies. For example, Sherif and Smith (1980) and Bhattacharyya and Fries (1982) considered a fatigue failure model in which accumulated decay is governed by a Wiener process. Doksum and Hoyland (1992) used a time-transformed Wiener process to model an accelerated degradation sample path. Doksum and Normand (1995) assumed biomarker processes such as calibrated log CD4 blood cell counts were Wiener processes in their HIV study. Whitmore and Schenkelberg (1997) also used a time-transformed Wiener process to model resistance of self-regulating heating cables. Elwany and Gebraeel (2009) used a Brownian motion with positive drift and the IG lifetime distribution to obtain a conservative estimate of an operating component's mean remaining life for subsequent managerial decisions. For other research efforts using the Wiener degradation model, see, for example, Whitmore, Crowder, and Lawless (1998), Padgett and Tomlinson (2004), Tseng, Tang, and Ku (2003), Tseng and Peng (2004), Balka, Desmond, and McNicholas (2009), Gebraeel, Lawley, Li, and Ryan (2005), Park and Padgett (2005), Lehmann (2009), and Wang (2010).

Before an ADT is conducted, one needs to decide how the stress level (stress loadings) should be increased or decreased. Several types of stress loading have been proposed in the literature including Constant-Stress ADT (CSADT) and Step-Stress ADT (SSADT). For a CSADT, Yu and Tseng (1998) proposed a stopping rule for terminating a degradation test. Park and Yum (1997) developed plans in which they determined the stress levels, the proportion of units allocated to each stress level, and the measurement times such that the asymptotic variance of the Maximum Likelihood Estimate (MLE) of the mean lifetime at the use condition is minimized. Typically, a test plan is designed so that a precise estimate can be obtained. For other references about CSADT, see Meeker and Escobar (1998), Yu (2003, 2006), Wu and Chang (2002), and Lim and Yum (2011).

Recently, researchers have considered using some time-varying stress loadings in order to further shorten test duration and reduce the number of test devices and the sample size. SSADT is a special type of stress loading in which all units are tested together and the stress level is increased step-wisely. When there are only two stress levels (i.e., one change), the test plan is often referred to as a *simple* SSADT plan. SSADT is commonly used because it is often easier to administer than a general time-varying stress plan and has the advantage that only a few test units are needed. It has been shown that using step-stress stress loading can provide equivalent estimation precision to that from other stress loadings (see Hu, Plante, & Tang, 2013; Liao & Elsayed, 2010). Further, Lee and Tang (2013) have shown that there exist SSADT plans that can generate a Fisher information matrix identical to that derived from a general stress loading function. Given these advantages of using SSADT, extensive studies have been conducted to obtain optimum SSADT plans. For example, Tang, Yang, and Xie (2004) designed a SSADT to minimize the total expected test cost, which is a function of sample size, test duration, and the number of inspections. Liao and Tseng (2006) provided SSADT plans to minimize the variance of estimated p-percentile under a budget constraint. Recently, Tseng et al. (2009) introduced a SSADT plan minimizing the approximate variance of the estimated MTTF when the degradation path follows a gamma process. Zhang, Jiang, Li, and Wang (2010) and Ge, Li, Jiang, and Huang (2011) have also provided algorithms to obtain SSADT plans for several different objectives. For an overview of degradation test models, as well as design problems, refer to Boulanger and Escobar (1994), Meeker and Escobar (1998), Nelson (2005a, 2005b), and Yum, Lim, and Seo (2007).

Although extensive research efforts, including the articles cited above, have been devoted to obtaining optimal SSADT plans, most of their results are based primarily on numerical studies. It is important to mention that, in many of these research efforts, the numerical results suggest that the optimum SSADT design is actually a simple SSADT using only the minimum and maximum stress levels even when their objectives are quite different. In this paper, we formally show that this result holds for many commonly used objective functions. Secondly, when focusing on designing a simple SSADT, the optimal allocations of inspection efforts are derived under various optimization criteria. The remainder of this paper is organized as follows. In Section 2, we describe the accelerated degradation model used in this paper and introduce the decision variables and constraints considered in designing a SSADT plan. In Section 3, we derive the MLEs of the model parameters and the Fisher information matrix. Using this matrix, we show that for several commonly used objective functions, the optimal SSADT plan is indeed a simple one. Optimal simple SSADT plans are then derived in Section 4. In Section 5, a numerical example is provided to compare the efficiency of a SSADT plan proposed by a previous study and the optimum plans proposed in this paper. A simulation study is also conducted for investigating the efficiency of optimum SSADT plans when the sample size is small. Finally, concluding remarks and possible directions for future study are given in Section 6.

2. The model and design for a SSADT

In this section, we introduce the notations and the model assumptions used throughout this paper. Decision variables, as well as constraints, for designing a SSADT plan are also discussed.

2.1. A SSADT plan

2.1.1. Decision variables

In a SSADT, all test units are exposed to an initial stress level (denoted by s_1) and tested independently until a pre-specified stress change time. The stress is then adjusted to another level (denoted by s_2) for the surviving test units. There may be more than one stress adjustment before the test is terminated. Under each stress level, s_i, the surviving units are inspected and the degradation increments are recorded at pre-specified time points $t_{ij}, j = 1, 2, \ldots, l_i, i = 1, 2, \ldots, k$ where k is the total number of stress levels and l_i is the number of inspections under s_i. We assume that under all stress levels, inspections are conducted at the same inspection time interval, Δt. Hence, the total test duration under s_i is $l_i \times \Delta t$. In this paper, we assume that the sample size (N), the inspection interval (Δt), and the stress levels ($s_i, i = 1, 2, \ldots, k$) are pre-specified and the decision variables are the number of inspections under each stress level (i.e., $l_i, i = 1, 2, \ldots, k$) when optimizing several of the commonly used objective functions described in Section 3.

2.1.2. Constraints in SSADT planning

To shorten the test duration, one of the most commonly used constraints is the time constraint. That is, a SSADT is terminated at a pre-specified time, since the total test budget is often limited in practice. The budget does affect not only the test duration but also the number of inspections. In this paper, we assume the total test time (T) and hence the total number of inspections (L) are constrained, where $T = \sum_{i=1}^{k} l_i \times \Delta t = L \times \Delta t$. An optimum SSADT plan is obtained by determining the number of inspections at each stress level to optimize a particular objective with the presence of the constraints.

2.2. Model assumptions

We first assume that there exists an upper stress bound, s_H, below which the failure mode is the same as the one under the normal use stress level, s_U. That is, the applied stress should not be too high so that the underlying failure mechanism may become different. For a multi-level SSADT using stress levels s_i, $i = 1, 2, \ldots, k$, define the standardized stress levels as

$$x_i = \frac{s_i - s_U}{s_H - s_U}, \quad \text{for } i = 1, 2, \ldots, k,$$

and hence the range of $x_i \in [0, 1]$, for all i.

We assume that the degradation path (possibly transformed) of a unit, under a constant stress level, say x_i, follows a stochastic Wiener process (denoted as $W_i(t)$) with drift and dispersion parameters $\eta_i > 0$ and $\sigma^2 > 0$, respectively:

$$W_i(t) = \eta_i t + \sigma B(t), \quad t \geq 0. \tag{1}$$

where $B(\cdot)$ is a standard Brownian motion. That is, $W_i(t)$ is expected to increase at a rate of η_i under x_i. Being a continuous-time version of the discrete-time cumulative sum (CUSUM) process, $W_i(t)$ is the solution for the stochastic linear growth model $dW_i(t) = \eta_i dt + \sigma dB(t)$.

Suppose there are N units available for a SSADT that uses stress levels $\{x_1, x_2, \ldots, x_k\}$ and the degradation path of each (surviving) unit is measured every Δt units of time. Because a Wiener process has independent increments and is memoryless, the degradation increments over a time interval depend only on the stress level over the interval and its length, not the past history of the path. Hence, the j-th ($j = 1, 2, \ldots, l_i$) degradation increment on $W_i(t)$ in (1) under stress level x_i for the h-th unit, y_{hij}, follows a normal distribution $N(\eta_i \Delta t, \sigma^2 \Delta t)$ independently with a Probability Density Function (PDF):

$$f(y_{hij}) = \frac{1}{\sqrt{2\pi\sigma^2\Delta t}} e^{-(y_{hij}-\eta_i\Delta t)^2/2\sigma^2\Delta t}, \quad \text{for all}$$
$$j = 1, 2, \ldots, l_i, \quad i = 1, 2, \ldots k, \quad \text{and} \quad h = 1, 2, \ldots N. \tag{2}$$

Further, we assume that the relationship between the standardized stress level x_i and the corresponding drift parameter is linear. The linear parameter–stress relationship is commonly used in many studies about Accelerated Degradation Test; for example, see Tang et al. (2004), Yu (2003) , and Lim and Yum (2011). Specifically, we assume

$$\eta_i = \alpha + \beta x_i, \tag{3}$$

where $(\alpha, \beta, \sigma^2)$ are unknown parameters to be estimated.

Under model (1), the product's lifetime under use condition ($x_0 = 0$), denoted by T_0, can be defined as the first-passage time of the degradation process ($W_0(t)$) over a constant threshold, a (>0), i.e.,

$$T_0 = \inf\{t \geq 0 | W_0(0) = 0; W_0(t) \geq a\}. \tag{4}$$

It is well-known that T_0 follows an inverse Gaussian distribution, denoted by IG(μ,λ), with location (i.e., mean) and scale parameters

$$\mu = \frac{a}{\alpha} \quad \text{and} \quad \lambda = \frac{a^2}{\sigma^2}. \tag{5}$$

The PDF and the Cumulative Distribution Function (CDF) of T_0 are

$$f(t) = \sqrt{\frac{\lambda}{2\pi t^3}} e^{-(\lambda(t-\mu)^2/2\mu^2 t)}, \quad t > 0, \tag{6}$$

$$F(t) = \Phi\left(\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\mu} - 1\right)\right) + e^{(2\lambda/\mu)}\Phi\left(-\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\mu} + 1\right)\right), \quad t > 0. \tag{7}$$

For a detailed introduction of the IG distribution, refer to Chhikara and Folks (1989) and Seshardi (1999). Based on the model assumptions in this section, we consider the optimum SSADT plans in the next section.

3. Fisher information matrix and optimization criteria

Most reliability measures such as MTTF and IG lifetime percentiles are functions of the model parameters, α, β, and σ. In this section, we obtain the MLEs of these parameters and derive the Fisher information matrix, which is normally used to quantify the information about the model and the parameters obtained from the data/experiment. We then describe several existing criteria, all based on the Fisher information matrix, for determining an optimum SSADT plan.

Based on assumptions (1), (2), and (3), the log-likelihood function of the parameters from the N test units is

$$l(\alpha, \beta, \sigma) = -\frac{NL\left(ln(2\pi) + ln(\sigma^2)\right)}{2} - \frac{N\sum_{i=1}^{k}(l_i \times ln(\Delta t))}{2} - \sum_{h=1}^{N}\sum_{i=1}^{k}\sum_{j=1}^{l_i}\frac{(y_{hij} - (\alpha + \beta x_i)\Delta t)^2}{2\sigma^2\Delta t} \tag{8}$$

where $L = \sum_{i=1}^{k} l_i$ is the total number of inspections. The MLEs of parameters $(\alpha, \beta, \sigma^2)$ can be obtained by setting the first partial derivatives of (8) with respect to each parameter to 0 and solving the equations jointly. We have

$$\hat{\alpha} = \frac{1}{NT}\left(\sum_{h=1}^{N}\sum_{i=1}^{k}\sum_{j=1}^{l_i} y_{hij} - \hat{\beta}N\Delta t\sum_{i=1}^{k} x_i l_i\right), \tag{9}$$

$$\hat{\beta} = \frac{\left(\sum_{h=1}^{N}\sum_{i=1}^{k}\sum_{j=1}^{l_i} y_{hij}\right)\left(\sum_{i=1}^{k} x_i l_i\right) - L\sum_{h=1}^{N}\sum_{i=1}^{k}\sum_{j=1}^{l_i} x_i y_{hij}}{N\Delta t\left[\left(\sum_{i=1}^{k} x_i l_i\right)^2 - L\sum_{i=1}^{k} x_i^2 l_i\right]}, \tag{10}$$

$$\hat{\sigma}^2 = \frac{1}{NL}\sum_{h=1}^{N}\sum_{i=1}^{k}\sum_{j=1}^{l_i}\frac{(y_{hij} - (\hat{\alpha} + \hat{\beta}x_i)\Delta t)^2}{\Delta t}. \tag{11}$$

In models that meet standard regularity conditions (including the model presented in this paper), the large-sample asymptotic variance–covariance matrix of these MLEs is the inverse of the Fisher information matrix (Ma & Meeker, 2008). The Fisher information matrix, $I(\alpha, \beta, \sigma)$, is obtained by taking the expected values of the negative second derivatives of the log-likelihood function in (8) with respect to the parameters, resulting in:

$$I(\alpha, \beta, \sigma) = \frac{N}{\sigma^2}\begin{bmatrix} \sum_{i=1}^{k} l_i\Delta t & \sum_{i=1}^{k} x_i l_i\Delta t & 0 \\ \sum_{i=1}^{k} x_i l_i\Delta t & \sum_{i=1}^{k} {x_i}^2 l_i\Delta t & 0 \\ 0 & 0 & 2\sum_{i=1}^{k} l_i \end{bmatrix}, \tag{12}$$

where N is the total sample size.

Since the asymptotic variance–covariance matrix of the MLEs is the inverse of the information matrix, many optimal SSADT design criteria are based on (12) if the goal of conducting an experiment is to estimate the model parameters (e.g., MTTF) or their functions (e.g., lifetime percentiles). We will show that the optimum multiple k-level SSADT plan will degenerate into a simple SSADT plan using only the minimum and the maximum stress levels for several optimization criteria. Before proceeding further, we briefly review the definitions of some commonly used criteria based on the discussion in Ng, Balakrishnan, and Chan (2007), as follows.

[C1] Maximize the determinant of the Fisher information matrix (*D-optimality*). D-optimality is one of the most popular design criteria used for designing an ALT/ADT. The asymptotic covariance matrix of the MLEs of the model parameters is proportional to the inverse matrix of the Fisher information matrix. As a univariate measure of overall variability, the determinant of this inverse matrix is often used and is called the generalized variance (Johnson & Wichern, 2007). Minimizing this determinant is equivalent to maximizing the determinant of the original Fisher information matrix, and hence the criterion of the D-optimality. Furthermore, the volume of a joint confidence region for all model parameters is inversely proportional to the square root of the determinant of the Fisher information matrix (Meeker & Escobar, 1998). Hence, by maximizing the determinant of the Fisher information matrix, we obtain the smallest volume for the Wald-type joint confidence region and thus the highest join precision of the model parameter estimators (Han & Ng, 2013). For more details, see Nalimov, Golikov, and Mikeshina (1970) and Montgomery (2013).

[C2] Minimize the asymptotic variance of the estimated MTTF at use condition, $x_0 = 0$. MTTF at use condition is an important measure of a product's reliability. Criterion **[C2]** focuses on the asymptotic variance of the MLE of this MTTF. In our case, the MTTF under x_0 is a/α (see (5)), the MLE is $a/\hat{\alpha}$, and follows a normal distribution, $N(a/\alpha, \mathrm{Var}(a/\hat{\alpha}))$, asymptotically. The asymptotic variance can be obtained by using the Delta-method (Meeker & Escobar, 1998), pages 619–620), i.e.,

$$\mathrm{Var}\left(\frac{a}{\hat{\alpha}}\right) = \left[-\frac{a}{\alpha^2} \quad 0 \quad 0\right] I^{-1}(\alpha,\beta,\sigma)\left[-\frac{a}{\alpha^2} \quad 0 \quad 0\right]'.$$

[C3] Minimize the trace of the first-order approximation of the variance–covariance matrix of the MLEs of the model parameters (*A*-optimality). Unlike **[C1]**, where one uses the determinant of the asymptotic covariance matrix of MLEs for the model parameters as a univariate measure of overall variability, **[C3]** uses the trace. Hence, A-optimality criterion minimizes the total variance of the parameter estimates, which is also the sum of the eigenvalues of the inverse of the Fisher information matrix (Han & Ng, 2013).

[C4] Minimize the asymptotic variance of the estimated p-th percentile ($\mathrm{Var}(\hat{t}_p)$) of the failure time distribution (i.e., the IG distribution) at use condition x_0. In addition to the MTTF in **[C2]**, percentiles of the product's lifetime distribution at use condition are other important reliability measures to be estimated (Liao & Tseng, 2006). Nelson (2005a, 2005b) also mentioned that, among various optimization criteria, this is the criterion used in most references when developing ALT plans. Using the Delta-method, the variance of the estimated p-th percentile can be expressed as (Lee & Tang, 2013)

$$\mathrm{Var}(\hat{t}_p) = \left[\frac{\partial t_p}{\partial\alpha} \; 0 \; \frac{\partial t_p}{\partial\sigma}\right] I^{-1}(\alpha,\beta,\sigma)\left[\frac{\partial t_p}{\partial\alpha} \; 0 \; \frac{\partial t_p}{\partial\sigma}\right]',$$

where

$$\frac{\partial t_p}{\partial\alpha} = \frac{2a}{\sigma^2} e^{2a\eta/\sigma^2}\Phi\left(-\sqrt{\frac{a^2}{t_p\sigma^2}}\left(\frac{\eta t_p}{a}+1\right)\right) f^{-1}(t_p),$$

$$\frac{\partial t_p}{\partial\sigma} = \left(\frac{2a}{\sigma^2\sqrt{t_p}}\phi\left(\sqrt{\frac{a^2}{t_p\sigma^2}}\left(\frac{\eta t_p}{a}-1\right)\right) - \frac{4a\eta}{\sigma^3}e^{2a\eta/\sigma^2}\Phi\left(-\sqrt{\frac{a^2}{t_p\sigma^2}}\left(\frac{\eta t_p}{a}+1\right)\right)\right) f^{-1}(t_p),$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the PDF and CDF of the standard normal distribution; and $f(\cdot)$ is the PDF of $\mathrm{IG}(a/\eta, a^2/\sigma^2)$ in (6).

For each of the four objectives, we investigate the optimum SSADT plan in the next section.

4. Optimum SSADT plan

In this section, we first show that for each criterion described in Section 3, a multi-level SSADT plan, when optimized, degenerates to a simple SSADT plan. This result establishes a rationale for considering a simple SSADT. Then, we consider the design problem of a simple SSADT plan and present the optimal allocation of inspections at each stress level for different criteria.

4.1. Degeneration of an optimum multi-level SSADT plan

In this section, by considering the decision variables $(l_1, l_2, \ldots, l_k)$, as well as the constraints (fixed Δt, L, and $T = L \times \Delta t$), we formally prove this observed phenomenon for the objective functions considered in Section 3. The following proposition summarizes our result:

Proposition 1. *If a product's quality characteristic follows the Wiener degradation process in (1) and the parameter–stress relationship is linear (satisfying (3)), a multi-level SSADT plan using stress levels $x_1 < x_2 < \cdots < x_k$ is reduced to a simple SSADT plan using only the minimum and maximum stress levels, x_1 and x_k, when optimized with respect to each of the criteria, **[C1]–[C4]**, given in Section 3.*

Before proceeding to proving this proposition, we first present the following lemma which will be used in the proof.

Lemma 1. *Let X be a discrete random variable with finite possible outcomes $(x_1, x_2, \ldots, x_k)$ where $x_1 < x_2 < \cdots < x_k$. For every probability allocation $(p_1, p_2, \ldots, p_k)$ where $p_i = Pr(X = x_i)$ for all $i = 1, 2, \ldots, k$ and $\sum_{i=1}^{k} p_i = 1$, one can find a probability allocation $(p_1', 0, \ldots, 0, p_k')$ (i.e., assigning probabilities to the maximum and minimum outcomes, x_1 and x_k) that results in the same expected value but the variance or the second moment of X using $(p_1', 0, \ldots, 0, p_k')$ is greater than or equal to the one with $(p_1, p_2, \ldots, p_k)$. The inequality is strict if any of $p_2, \ldots, p_{k-1}$ is not zero.*

Proof. For a given $(p_1, p_2, \ldots, p_k)$, define $(p_1', p_k') = (p_1 + \sum_{j=2}^{k-1} p_j q_j, p_k + \sum_{j=2}^{k-1} p_j(1-q_j))$ where $0 < q_j < 1$ is the percentage satisfying $x_j = x_1 q_j + x_k(1-q_j)$, for $j = 2, \ldots k-1$. The existence of such q_j's is guaranteed because we assume $x_1 < x_2 < \cdots < x_k$. Since the square function is strictly convex, we have $x_j^2 < x_1^2 q_j + x_k^2(1-q_j)$, for $j = 2, \ldots k-1$, by Jensen's inequality, and consequently

$$x_1 p_1' + x_k p_k' = x_1\left(p_1 + \sum_{j=2}^{k-1} p_j q_j\right) + x_k\left(p_k + \sum_{j=2}^{k-1} p_j(1-q_j)\right) = \sum_{i=1}^{k} x_i p_i,$$

$$\sum_{i=1}^{k} x_i^2 p_i \le x_1^2 p_1 + \left(\sum_{j=2}^{k-1} p_j(x_1^2 q_j + x_k^2(1-q_j))\right) + x_k^2 p_k = x_1^2 p_1' + x_k^2 p_k'.$$

Therefore, $(p_1, p_2, \ldots, p_{k-1}, p_k)$ and $(p'_1, 0, \ldots, 0, p'_k)$ have the same expected value but the second moment of X (and thus the variance) using $(p'_1, 0, \ldots, 0, p'_k)$ is greater than or equal to the one with $(p_1, p_2, \ldots, p_k)$ as claimed. The inequality above is strict if any of $p_2, \ldots, p_{k-1}$ is not zero.

Based on Lemma 1, we have the following corollary.

Corollary 1. *For a positive discrete random variable X with finite possible outcomes* $(x_1, x_2, \ldots, x_k)$ *where* $0 < x_1 < x_2 < \ldots < x_k$, *to maximize* $E(X^2)/E(X)$, $Var(X)/E(X)$, or $\sqrt{Var(X)}/E(X)$ *(the coefficient of variation), one should assign probabilities to* x_1 *and* x_k *only.*

The proof of Corollary 1 follows the conclusion from Lemma 1 directly, and, therefore, it is omitted here. Lemma 1 and Corollary 1 give a similar conclusion regarding the allocation of probabilities to maximize certain variation measurements. That is, to maximize either the variance or the coefficient of variation of a discrete random variable, one should assign probabilities to the minimum and maximum outcomes. In our original non-linear optimization problem of designing a SSADT plan, we consider k decision variables $(l_1, l_2, \ldots, l_k)$ with $\sum_{i=1}^{k} l_i = L$. However, when the total number of inspections (L) is fixed, the original problem can be transformed by considering the proportions of inspections under each stress level, $l_1/L,\ l_2/L, \ldots,\ l_k/L$. These inspection proportions will be treated as probabilities in Lemma 1. Based on these results, we now present the proof of Proposition 1.

Proof of Proposition 1. For **[C1]**: For fixed N, $L = \sum_{i=1}^{k} l_i$, and the censoring time, $T = \sum_{i=1}^{k} l_i \Delta t$, the determinant of matrix (12) is

$$\frac{2N^3L^3\Delta t^2}{\sigma^6}\left(\sum_{i=1}^{k} x_i^2\left(\frac{l_i}{L}\right) - \left(\sum_{i=1}^{k} x_i\left(\frac{l_i}{L}\right)\right)^2\right),$$

where l_i/L is the relative inspection frequency under stress level x_i.

Define a discrete random variable$X$ with finite possible outcomes x_i and let $p_i \equiv \Pr(X = x_i) = l_i/L$ for $i = 1, 2, \ldots, k$. When N, L, and Δt are fixed, the determinant above is proportional to the variance of X. The original problem is then to maximize the variance of X. Thus, by Lemma 1, for an arbitrary given allocation of inspections $(l_1, l_2, \ldots, l_k)$, there exists another allocation (denoted as $(l'_1, 0, \ldots, 0, l'_k)$) that yields same expected value but equal or larger variance. Therefore, we have

$$\left(\sum_{i=1}^{k} x_i^2\left(\frac{l_i}{L}\right) - \left(\sum_{i=1}^{k} x_i\left(\frac{l_i}{L}\right)\right)^2\right)$$
$$\le \left(\left(x_1^2\left(\frac{l'_1}{L}\right) + x_k^2\left(\frac{l'_k}{L}\right)\right) - \left(x_1\left(\frac{l'_1}{L}\right) + x_k\left(\frac{l'_k}{L}\right)\right)^2\right).$$

The inequality above is strict if any of $(l_1, l_2, \ldots, l_k)$ is not zero. That is, any multi-level, non-simple SSADT cannot be optimal. Thus, the optimum SSADT plan is a simple SSADT as claimed.

For **[C2]**: The inverse of (12) is

$$\frac{\sigma^2}{N}\begin{bmatrix} \dfrac{\sum_{i=1}^{k} x_i^2 l_i \Delta t}{\left(\sum_{i=1}^{k} l_i \Delta t\right)\left(\sum_{i=1}^{k} x_i^2 l_i \Delta t\right) - \left(\sum_{i=1}^{k} x_i l_i \Delta t\right)^2} & \dfrac{-\sum_{i=1}^{k} x_i l_i \Delta t}{\left(\sum_{i=1}^{k} l_i \Delta t\right)\left(\sum_{i=1}^{k} x_i^2 l_i \Delta t\right) - \left(\sum_{i=1}^{k} x_i l_i \Delta t\right)^2} & 0 \\ \dfrac{-\sum_{i=1}^{k} x_i l_i \Delta t}{\left(\sum_{i=1}^{k} l_i \Delta t\right)\left(\sum_{i=1}^{k} x_i^2 l_i \Delta t\right) - \left(\sum_{i=1}^{k} x_i l_i \Delta t\right)^2} & \dfrac{\sum_{i=1}^{k} l_i \Delta t}{\left(\sum_{i=1}^{k} l_i \Delta t\right)\left(\sum_{i=1}^{k} x_i^2 l_i \Delta t\right) - \left(\sum_{i=1}^{k} x_i l_i \Delta t\right)^2} & 0 \\ 0 & 0 & \dfrac{1}{2\sum_{i=1}^{k} l_i} \end{bmatrix}$$

so the asymptotic variance of the MLE of MTTF is

$$\frac{a^2\sigma^2}{\alpha^4 N \Delta t L}\left(\frac{\sum_{i=1}^{k} x_i^2\,(l_i/L)}{\left(\sum_{i=1}^{k} x_i^2\,(l_i/L)\right) - \left(\sum_{i=1}^{k} x_i\,(l_i/L)\right)^2}\right).$$

Hence, when N, L, and Δt are fixed, minimizing this variance is equivalent to maximizing

$$1 - \frac{\left(\sum_{i=1}^{k} x_i\,(l_i/L)\right)^2}{\sum_{i=1}^{k} x_i^2\,(l_i/L)}.$$

Consider the discrete random variable X as defined in the previous proof. The optimization problem is to maximize (by choosing the allocation of inspections)

$$1 - \frac{E(X)^2}{E(X^2)}.$$

For an arbitrary given allocation of inspections $(l_1, l_2, \ldots, l_k)$, by Lemma 1, we can always find another allocation (denoted as $(l'_1, 0, \ldots, 0, l'_k)$) using the minimum and maximum stresses that yields the same expected value (say c) but the variance or second moment is greater than or equal to the one with $(l_1, l_2, \ldots, l_k)$. Therefore,

$$\left.\left(1 - \frac{c^2}{E(X^2)}\right)\right|_{(l_1/L, l_2/L, \ldots, l_k/L)} \le \left.\left(1 - \frac{c^2}{E(X^2)}\right)\right|_{(l'_1/L, 0, \ldots, 0, l'_k/L)}.$$

Furthermore, according to Lemma 1, the inequality above is strict if any of $l_2, \ldots, l_{k-1}$ is non-zero. That is, any multi-level, non-simple SSADT can be improved. Thus, a simple SSADT is optimal, as claimed.

For **[C3]**: Based on the variance–covariance matrix given in the proof for [C2], the criterion [C3] is to minimize

$$\frac{\sigma^2}{NL}\left(\frac{1}{\Delta t}\,\frac{\sum_{i=1}^{k} x_i^2\,(l_i/L) + 1}{\left(\sum_{i=1}^{k} x_i^2\,(l_i/L)\right) - \left(\sum_{i=1}^{k} x_i\,(l_i/L)\right)^2} + \frac{1}{2}\right).$$

Following the same approach, the current optimization problem is to choose an allocation that maximizes

$$\frac{1 - (E(X))^2/E(X^2)}{1 + 1/E(X^2)}.$$

For an arbitrary allocation $(l_1, l_2, \ldots, l_k)$, we can always find $(l'_1, 0, \ldots, 0, l'_k)$ which has an identical expected value (without loss of generality, assume $E(X) = c$) but has equal or larger $E(X^2)$ value as shown in the previous proof. Hence, for this objective function, we have

$$\left.\frac{1 - c^2/E(X^2)}{1 + 1/E(X^2)}\right|_{(l_1/L, l_2/L, \ldots, l_k/L)} \le \left.\frac{1 - c^2/E(X^2)}{1 + 1/E(X^2)}\right|_{(l'_1/L, 0, \ldots, 0, l'_k/L)}.$$

Therefore, for any allocation of inspections, the objective function value is larger than or equal to the optimum simple SSADT plan when using the minimum and maximum stresses. The optimum SSADT plan is thus a simple SSADT as claimed.

For **[C4]**: Based on the variance–covariance matrix given in the proof for [C2], the criterion [C4] is to minimize

$$\mathrm{Var}(\hat{t}_p) = \frac{\sigma^2}{NL} \times \left(\left(\frac{\partial t_p}{\partial \alpha}\right)^2 \frac{1}{\Delta t} \frac{\sum_{i=1}^{k} x_i^2 (l_i/L)}{\left[\left(\sum_{i=1}^{k} x_i^2 (l_i/L)\right) - \left(\sum_{i=1}^{k} x_i (l_i/L)\right)^2\right]} + \frac{1}{2}\left(\frac{\partial t_p}{\partial \sigma}\right)^2 \right).$$

Because both $\partial t_p/\partial\alpha$ and $\partial t_p/\partial\sigma$ do not depend on l_i's, following the same approach used in the previous proofs, the current optimization problem involves choosing an allocation that minimizes

$$\frac{\sum_{i=1}^{k} x_i^2 (l_i/L)}{\left(\sum_{i=1}^{k} x_i^2 (l_i/L)\right) - \left(\sum_{i=1}^{k} x_i (l_i/L)\right)^2}.$$

Hence, based on the discussion in the proof for [C2], the optimum SSADT plan is a simple SSADT that uses only the minimum and maximum stresses under criterion [C4]. □

Remark. The results in Proposition 1 can be extended to cases where the relationship between the drift level and the standardized stress level follows a general linear model, i.e., $\eta_i = \alpha + \beta g(x_i)$, for some monotone function $g(\cdot)$. This can be shown by renaming $g(x_i)$ as x_i' and our proof of the proposition follows for these new x_i' s.

Notice that Lemma 1 together with the Proposition 1 provide an important insight regarding the optimum design of a SSADT plan. That is, a SSADT plan using more than two distinct stress levels or a simple SSADT plan using any of the intermediate (i.e., non-maximum or non-minimum) stress levels cannot be optimal. This observation is useful for searching for an optimum SSADT, since one needs to focus only on simple SSADTs using the minimum and maximum stress levels. Based on this observation, we derive the optimal allocation of inspection efforts in a simple SSADT plan in the next section.

4.2. Optimal simple SSADT plan

In Section 4.1, we have shown in Proposition 1 that for several commonly used optimization criteria, the optimum multi-level SSADT plan is a simple SSADT using the minimum and maximum stress levels. In what follows, we provide the optimal allocation of inspections at each stress level in a simple SSADT plan for each criterion.

When using only the minimum and maximum stress levels (x_1 and x_k), the information matrix (12) becomes

$$I(\alpha,\beta,\sigma) = \frac{N}{\sigma^2}\begin{bmatrix} l_1\Delta t + l_k\Delta t & x_1 l_1\Delta t + x_k l_k \Delta t & 0 \\ x_1 l_1 \Delta t + x_k l_k \Delta t & x_1{}^2 l_1 \Delta t + x_k{}^2 l_k \Delta t & 0 \\ 0 & 0 & 2(l_1 + l_k) \end{bmatrix}.$$

The only decision variable is the number of inspections under x_1 (i.e., l_1) since we assume the total number of inspections is pre-fixed (L) (so $l_k = L - l_1$ if l_1 is decided). The optimum allocation for each criterion is given in the following proposition.

Proposition 2. *For each of the criteria proposed in Section 3, the optimum SSADT plan is given as follows:*

(i) *For **[C1]**, the optimum plan assigns inspections:* $l_1/L = l_k/L = 1/2$.
(ii) *For **[C2]** and **[C4]**, the optimum plan assigns inspections:* $l_1/L = 1 - l_k/L = x_k/(x_1 + x_k)$.
(iii) *For **[C3]**, the optimum plan assigns inspections:* $l_1/L = 1 - l_k/L = \frac{(x_k{}^2+1)-\sqrt{(x_k{}^2+1)(x_1{}^2+1)}}{x_k{}^2-x_1{}^2}$.

Proof. For (i): By the proof of Proposition 1 for [C1] and Lemma 1, to objective function for criterion [C1] is to maximize $(x_1{}^2(l_1/L) + x_k{}^2(l_k/L)) - (x_1(l_1/L) + x_k(l_k/L))^2$. To facilitate our discussion, let $p_1 = l_1/L = 1 - l_k/L$. By taking the first derivative of the objective function with respect to p_1 and setting the results zero, it is straightforward to show that the maximum determinant of the information matrix is reached by setting the proportions as $l_1/L = l_k/L = 1/2$. This conclusion is consistent with the lemma proposed by Murthy and Sethi (1965). The optimum simple SSADT plan thus commits half of the inspection efforts at each stress level.

For (ii): As discussed in proofs of Proposition 1 for [C2] and [C4], the objective function is to maximize

$$1 - \frac{(x_1(l_1/L) + x_k(l_k/L))^2}{x_1{}^2(l_1/L) + x_k{}^2(l_k/L)}.$$

To facilitate our discussion, let $p_1 = l_1/L = 1 - l_k/L$. The objective function becomes

$$1 - \frac{(x_1 p_1 + x_k(1-p_1))^2}{x_1{}^2 p_1 + x_k{}^2(1-p_1)}.$$

Take the first derivative with respect to p_1 and let the result equal to 0, it is straightforward to show that the optimal value of l_1/L is $x_k/(x_1 + x_k)$.

For (iii): As discussed in the proof of Proposition 1 for [C3], the objective function is to maximize

$$\frac{(x_1^2 p_1 + x_k^2(1-p_1)) - (x_1 p_1 + x_k(1-p_1))^2}{(x_1^2 p_1 + x_k^2(1-p_1)) + 1},$$

where $p_1 = l_1/L$.

Similarly, by taking the first derivative with respect to p_1 and letting the result be equal to 0, we obtain $\hat{p}_1 = \frac{(x_k{}^2+1)-\sqrt{(x_k{}^2+1)(x_1{}^2+1)}}{x_k{}^2-x_1{}^2}$. Since $x_1 < x_k$, the obtained $\hat{p}_1$ is always feasible because

$$0 = \frac{(x_k{}^2+1) - \sqrt{(x_k{}^2+1)(x_k{}^2+1)}}{x_k{}^2 - x_1{}^2} < \frac{(x_k{}^2+1) - \sqrt{(x_k{}^2+1)(x_1{}^2+1)}}{x_k{}^2 - x_1{}^2} < \frac{(x_k{}^2+1) - \sqrt{(x_1{}^2+1)(x_1{}^2+1)}}{x_k{}^2 - x_1{}^2} = 1.$$ □

Based on Propositions 1 and 2, a simple SSADT plan uses the optimal allocation of inspections could generate the most efficient statistical results. However, note that this research does not intend to suggest that a simple SSADT is the best and only choice for designing a degradation experiment. An experimenter may need to use more than two stress levels in order to verify whether the linear relationship in (3) is valid. At times it could be beneficial to use more than two stress levels to attain more flexibility with some loss of efficiency.

In the next section, we study a numerical example for comparisons of efficiency between an existing multiple steps SSADT and the optimal simple SSADT.

5. A numerical example

Tseng and Wen (2000) proposed a numerical example analyzing the degradation of a certain type of LED lamps. LED lamps are a key component in contact image sensors used in fax machines, document scanners, copy machines, mark readers, and other office automation equipments. The lifetime of an LED lamp is highly correlated with its light intensity (brightness). However, the light intensity degrades very slowly. Thus, it is difficult to use an ALT to assess the product's lifetime with few test units and limited test duration. To overcome this issue, a SSADT with a time constraint was conducted to collect timely degradation data for accessing the reliability of this product. In Tseng and Wen's study, temperature was chosen as the accelerated variable and five different levels of temperature were used in the experiment. The used temperatures and the temperature change times are as follows:

$$\text{temperature} = \begin{cases} \text{25 degrees Celsius} & \text{when } 0 \le t < 1104 \text{ hours.} \\ \text{45 degrees Celsius} & \text{when } 1104 \le t < 3120 \text{ hours.} \\ \text{65 degrees Celsius} & \text{when } 3120 \le t < 5808 \text{ hours.} \\ \text{85 degrees Celsius} & \text{when } 5808 \le t < 8110 \text{ hours.} \\ \text{105 degrees Celsius} & \text{when } 8110 \le t < 9118 \text{ hours.} \end{cases}$$

In this experiment, 22 LED lamps were tested under this SSADT plan and the light intensities were measured every 168 hours except for the first interval [0, 1104). Following the assumptions used in Tseng and Wen (2000), we have $s_U = 25$ degrees Celsius (use condition), $s_H = 105$ degrees Celsius (highest test temperature), and $N = 22$. Therefore, the standardized stresses are

$$\text{standardized temperature} = \begin{cases} x_1 = 0.00 & \text{when } s_1 = 25 \text{ degrees Celsius;} \\ x_2 = 0.25 & \text{when } s_2 = 45 \text{ degrees Celsius;} \\ x_3 = 0.50 & \text{when } s_3 = 65 \text{ degrees Celsius;} \\ x_4 = 0.75 & \text{when } s_4 = 85 \text{ degrees Celsius;} \\ x_5 = 1.00 & \text{when } s_5 = 105 \text{ degrees Celsius.} \end{cases}$$

Moreover, their allocation of inspections in this SSADT plan was

$$(l_1, l_2, l_3, l_4, l_5) \approx (6.57, 12, 16, 13.70, 6).$$

Some of these numbers are not integers because the original test duration at each stress level may not be a multiple of 168 hours. As an illustrative example, we modify the original SSADT plan by rounding up the non-integer values and consider $(l_1, l_2, l_3, l_4, l_5) = (7, 12, 16, 14, 6), L = \sum_{i=1}^{5} l_i = 55$. Later, results from the optimal design of SSADT are compared with those from this modified SSADT plan. For more detailed experimental settings, as well as the graph of the degradation paths of the 22 LED lamps, refer to Tseng and Wen (2000).

5.1. Optimum step-stress accelerated degradation plans

Because the original five steps SSADT plan in Tseng and Wen (2000) was not statistically optimal, Liao and Tseng (2006) obtained optimal SSADT plans by minimizing the variance of the estimated p-th percentile subject to a test budget constraint. They used a negative log and a time scale transformation of the original standardized degradation paths so that the transformed paths would follow the Wiener/IG distribution model as described in Section 2. For the parameter–stress relationship, they considered the Arrhenius reaction rate model and the estimated relationship between η_i and the temperature stress s_i to be

$$\hat{\eta}_i = \exp\left(5.3669 + \frac{-2546.7}{273.16 + s_i}\right),$$

and the MLE of σ^2 to be 0.00082, as obtained from the data in Tseng and Wen (2000). Another commonly used parameter–stress relationship is the linear relation described in (3) (see Tang et al., 2004; Yu, 2003). To compare both relationships and stay consistent with our assumption, we fitted a simple linear regression line for the estimated $\hat{\eta}_i$ from the Arrhenius reaction rate model and the standardized stress levels x_i within the range of the test temperature (from 25 degrees Celsiusto105 degrees Celsius) and obtained:

$$\hat{\eta}_i = 0.0212 + 0.2096 x_i.$$

The coefficient of determination, R^2, is 97.3 percent , which suggests that within the given range of temperature in this example, the difference between the two models is not significant; they both provide a reasonable fit to the parameter–stress relationship and (3) may be easier to interpret from a practical point of view.

From the fitted simple regression line and following the transformation used in Liao and Tseng (2006), we assume that the parameters in the model described in Section 2 are $(\hat{\alpha}, \hat{\beta}, \hat{\sigma}^2) = (0.0212, 0.2096, 0.00082)$ and $\Delta t = 4.26$. Furthermore, to calculate variances of the estimated MTTF and p-th percentile of the IG distribution under use condition, it is necessary to specify the value of the threshold (a) so that the product is considered failed when the Wiener degradation path crosses this threshold. Usually, the threshold for the brightness of an LED lamp is specified as 50 percent of its original brightness (Lee & Tang, 2007). Therefore, after a negative log transformation, the threshold value for the assumed Wiener/IG distribution model is $-ln(0.5) = 0.693147$.

Suppose 22 LED lamps are available for testing, the previous five stress levels are to be used when conducting a SSADT and there are 55 inspections to be allocated into each stress level. Then, according to Proposition 2 in Section 4, the optimal allocations of inspections are: for [C1], we have $(l'_1, l'_5) = (27.5, 27.5)$; for [C2] and [C4], we have $(l'_1, l'_5) = (55, 0)$; and for [C3], we have $(l'_1, l'_5) = (32.22, 22.78)$. Note that these numbers are calculated by multiplying the theoretical optimal proportions by 55 and thus the resulting allocations might not be integers. Therefore, we use the nearest integers and require that at least 20 percent of inspections be taken at both s_1 and s_k. For example, if one would like to maximize the determinant of the Fisher information matrix (under [C1]), we would conduct (27, 28) or (28, 27) inspections at stress levels (25 degrees Celsius, 105 degrees Celsius), respectively. If the goal is to estimate the MTTF or the 10th percentile of the IG distribution at 25 degrees Celsius ([C2] and [C4]), then we will allocate 44 inspections at 25 degrees Celsius and 11 inspections at 105 degrees Celsius. Finally, if the goal is to minimize the trace of the variance–covariance matrix (under [C3]), then the optimal SSADT plan will have (32, 23) or (33, 22) inspections at stress levels (25 degrees Celsius, 105 degrees Celsius), respectively.

To compare any given SSADT plans with the above optimal simple SSADT plan, we calculated the values of the objective function for each SSADT plan. In addition, we also computed the Relative Efficiency (RE) of a SSADT plan to the corresponding optimal SSADT plans under [C1] to [C4] as defined in Ng et al. (2007):

$$\mathrm{RE}(l_1, \ldots, l_5) = \frac{\det(\mathbf{I}) \text{ corresponding to the SSADT plan} (l_1, \ldots, l_5)}{\det(\mathbf{I}) \text{ corresponding to the optimal SSADT plan for [C1]}},$$

$$\mathrm{RE}(l_1, \ldots, l_5) = \frac{\mathrm{Var}(a/\hat{\alpha}) \text{ corresponding to the optimal SSADT plan for [C2]}}{\mathrm{Var}(a/\hat{\alpha}) \text{ corresponding to the SSADT plan } (l_1, \ldots, l_5)},$$

$$\mathrm{RE}(l_1, \ldots, l_5) = \frac{\mathrm{Trace}(\mathbf{I}^{-1}) \text{ corresponding to the optimal SSADT plan for [C3]}}{\mathrm{Trace}(\mathbf{I}^{-1}) \text{ corresponding to the SSADT plan} (l_1, \ldots, l_5)},$$

and

$$\mathrm{RE}(l_1, \ldots, l_5) = \frac{\mathrm{Var}(\hat{t}_{0.1}) \text{ corresponding to the optimal SSADT plan for [C4]}}{\mathrm{Var}(\hat{t}_{0.1}) \text{ corresponding to the SSADT plan } (l_1, \ldots, l_5)}.$$

Table 1 presents the results of comparisons between the optimal SSADT plans and the original SSADT plan of Tseng and Wen (2000).

For both [C1] and [C3], we rounded the optimal allocation of inspections to the two nearest integer solutions. Notice that the proposed optimum SSADT plans have higher efficiency than the plan from Tseng and Wen (2000) under all criteria. For criterion [C4], the improvement in efficiency is not very significant. However, in the most extreme case, the optimum plan could improve the efficiency by 67 percent (for [C2]). We also notice that the SSADT plan for [C3] in Table 1 not only optimizes the criterion [C3] but also provides high

Table 1
Comparisons between the SSADT plan used in Tseng and Wen (2000) and the optimal plans.

	$(l_1, l_2, l_3, l_4, l_5)$	Det(I) (RE)	Var$(a/\hat{\alpha})$ (RE)	Trace(I^{-1}) (RE)	Var$(\hat{t}_{0.1})$ (RE)
Original plan	(7, 12, 16, 14, 6)	1.0337×10^{19} (0.3547)	1.4428 (0.3272)	2.7413×10^{-6} (0.4618)	0.0333 (0.9054)
[1]	(27, 0, 0, 0, 28)	2.9147×10^{19} (1.0000)	0.7693 (0.6136)	1.2994×10^{-6} (0.9743)	0.0311 (0.9690)
	(28, 0, 0, 0, 27)	2.9147×10^{19} (1.0000)	0.7418 (0.6364)	1.2878×10^{-6} (0.9831)	0.0311 (0.9718)
[2] and [4]	(44, 0, 0, 0, 11)	1.8660×10^{19} (0.6402)	0.4721 (1.0000)	1.5319×10^{-6} (0.8265)	0.0302 (1.0000)
[3]	(32, 0, 0, 0, 23)	2.8376×10^{19} (0.9735)	0.6491 (0.7273)	1.2661×10^{-6} (1.0000)	0.0308 (0.9813)
	(33, 0, 0, 0, 22)	2.7991×10^{19} (0.9603)	0.6294 (0.7500)	1.2668×10^{-6} (0.9994)	0.0307 (0.9833)

Note: [*i*] implies the optimal SSADT plan for $[\mathrm{C}_i]$ where $i = 1, 2, 3,$ and 4. RE refers to the relative efficiency between the proposed plan and the corresponding optimal plan.

RE for other criteria. It may be practically beneficial to use this kind of *robust* plan that could provide reasonable high RE for many optimization criteria at the same time when one is not only interested in a particular objective function. In addition, even some SSADT plans in Table 1 are obtained by rounding the theoretical optimal numbers of inspections to their nearest integer values, this rounding does not affect the efficiency significantly. Based on the results in Table 1, an experimenter could gain significant improvement in efficiency by using the optimal number of inspections.

5.2. Simulation study for small sample size

The objective functions, as well as the corresponding optimum SSADT plans, are all based on the asymptotic variance–covariance matrix under the assumption of a large sample size. Therefore, it is necessary to investigate the performance of the proposed optimal SSADT designs when the sample size is only moderate or even small (which often occurs for newly developed products). A simulation study is conducted here to demonstrate that the SSADT designs presented earlier can also attain better performance than other SSADT plans when the sample size is small.

To simulate random Wiener degradation paths for this study, the following experimental settings are used: $N = 3$, $k = 5$, $(x_1, x_2, x_3, x_4, x_5) = (0, 1/4, 2/4, 3/4, 1)$, $(\alpha, \beta, \sigma^2) = (0.02121, 0.2096, 0.00082)$, $a = 0.6931$, and the total number of inspection (L)= 10, 20, and 30 are used to represent small sample size scenarios. For each value of L, we obtain the optimum SSADT plans according to Proposition 2 and compare their results to an equally allocated SSADT plan. When the theoretical allocation is not an integer, we round it to the nearest integer as above. For each SSADT plan, we simulate 1000 degradation data sets, each with $N = 3$ degradation paths, and then calculate the MLEs of model parameters, as well as

Table 2
Theoretical asymptotic results for each SSADT plan under different sample sizes.

L	$(l_1, l_2, l_3, l_4, l_5)$	Theoretical asymptotic results			
		Det(I) (RE)	Var$(a/\hat{\alpha})$(RE) 45.3524 (0.4167)	Trace(I^{-1}) (RE)	Var$(\hat{t}_{0.1})$ (RE)
10	(2, 2, 2, 2, 2) [equally allocated]	2.2219×10^{14} (0.5000)	45.3524 (0.4167)	8.4243×10^{-5} (0.6065)	1.3021 (0.9335)
	(5, 0, 0, 0, 5) [1]	4.4438×10^{14} (1.0000)	30.2349 (0.6250)	5.2163×10^{-5} (0.9795)	1.2527 (0.9704)
	(8, 0, 0, 0, 2) [2] and [4]	2.8440×10^{14} (0.6400)	18.8968 (1.0000)	6.1787×10^{-5} (0.8269)	1.2156 (1.0000)
	(6, 0, 0, 0, 4) [3]	4.2660×10^{14} (0.9600)	25.1958 (0.7500)	5.1093×10^{-5} (1.0000)	1.2362 (0.9833)
20	(4, 4, 4, 4, 4) [equally allocated]	1.7775×10^{15} (0.5000)	22.6762 (0.4167)	4.2121×10^{-5} (0.6065)	0.6511 (0.9335)
	(10, 0, 0, 0, 10) [1]	3.5550×10^{15} (1.0000)	15.1175 (0.6250)	2.6081×10^{-5} (0.9795)	0.6263 (0.9704)
	(16, 0, 0, 0, 4) [2] and [4]	2.2752×10^{15}(0.6400)	9.4484 (1.0000)	3.0893×10^{-5} (0.8269)	0.6078 (1.0000)
	(12, 0, 0, 0, 8) [3]	3.4128×10^{15} (0.9600)	12.5979 (0.7500)	2.5547×10^{-5} (1.0000)	0.66181 (0.9833)
30	(6, 6, 6, 6, 6) [equally allocated]	0.5999×10^{16} (0.5000)	15.1175 (0.4167)	2.8081×10^{-5} (0.6065)	0.4340 (0.9335)
	(15, 0, 0, 0, 15) [1]	1.1998×10^{16} (1.0000)	10.0783 (0.6250)	1.7388×10^{-5} (0.9795)	0.4176 (0.9704)
	(24, 0, 0, 0, 6) [2] and [4]	0.7679×10^{16} (0.6400)	6.2989 (1.0000)	2.0596×10^{-5} (0.8269)	0.4052 (1.0000)
	(18, 0, 0, 0, 12) [3]	1.1518×10^{16} (0.9600)	8.3986 (0.7500)	1.7031×10^{-5} (1.0000)	0.4121 (0.9833)

Note: [*i*] implies the theoretical optimal SSADT plan for $[\mathrm{C}_i]$ where $i = 1, 2, 3,$ and 4.

Table 3
The average of 1000 simulated results from estimated Fisher information matrix for each SSADT plan under different sample sizes.

L	$(l_1, l_2, l_3, l_4, l_5)$	The average of each objective function from 1000 simulated degradation paths			
		Det(I) (RE)	Var$(a/\hat{\alpha})$(RE)	Trace(I^{-1}) (RE)	Var$(\hat{t}_{0.1})$ (RE)
10	(2, 2, 2, 2, 2) [equally allocated]	4.4187×10^{14} (0.4746)	168.2340 (0.1272)	7.7536×10^{-5} (0.6097)	1.3068 (0.9298)
	(5, 0, 0, 0, 5) [1]	9.3113×10^{14} (1.0000)	41.1620 (0.5198)	4.8357×10^{-5} (0.9775)	1.2533 (0.9695)
	(8, 0, 0, 0, 2) [2] and [4]	5.5262×10^{14} (0.5935)	21.3960 (1.0000)	5.7905×10^{-5} (0.8164)	1.2151 (1.0000)
	(6, 0, 0, 0, 4) [3]	8.6273×10^{14} (0.9265)	30.6692 (0.6976)	4.7271×10^{-5} (1.0000)	1.2379 (0.9816)
20	(4, 4, 4, 4, 4) [equally allocated]	2.5004×10^{15} (0.5268)	27.6857 (0.3647)	4.0483×10^{-5} (0.6133)	0.6510 (0.9353)
	(10, 0, 0, 0, 10) [1]	4.7463×10^{15} (1.0000)	16.6390 (0.6068)	2.5361×10^{-5} (0.9789)	0.6244 (0.9752)
	(16, 0, 0, 0, 4) [2] and [4]	3.1459×10^{15} (0.6628)	10.0972 (1.0000)	2.9786×10^{-5} (0.8335)	0.6089 (1.0000)
	(12, 0, 0, 0, 8) [3]	4.5689×10^{15} (0.9626)	13.8773 (0.7276)	2.4827×10^{-5} (1.0000)	0.6184 (0.9846)
30	(6, 6, 6, 6, 6) [equally allocated]	0.7376×10^{16} (0.5000)	16.8068 (0.3862)	2.7528×10^{-5} (0.6040)	0.4327 (0.9360)
	(15, 0, 0, 0, 15) [1]	1.4753×10^{16} (1.0000)	10.8789 (0.5966)	1.6997×10^{-5} (0.9782)	0.4175 (0.9701)
	(24, 0, 0, 0, 6) [2] and [4]	0.9550×10^{16} (0.6473)	6.4904 (1.0000)	2.0088×10^{-5} (0.8277)	0.4050 (1.0000)
	(18, 0, 0, 0, 12) [3]	1.4166×10^{16} (0.9602)	8.9006 (0.7292)	1.6627×10^{-5} (1.0000)	0.4125 (0.9818)

Note: [*i*] implies the theoretical optimal SSADT plan for $[\mathrm{C}_i]$ where $i = 1, 2, 3,$ and 4.

the estimated Fisher information matrix for each data set. The asymptotic results obtained by using matrix (12) and the average of 1000 simulated objective function values are presented in Tables 2 and 3, respectively.

The simulation results in Tables 2 and 3 show that the proposed optimal SSADT plans presented earlier continue to hold for relatively small sample sizes. For criteria [C3] and [C4], it seems the choice of the SSADT plan is not crucial since all plans generate high or nearly full efficiency. However, criteria [C1] and [C2] are relatively sensitive to the plan used because the optimum SSADT plans have significantly higher efficiency than do other plans.

6. Summary

In this paper, a Wiener process/IG distribution model is assumed to describe the accelerated degradation path of certain quality characteristics and the lifetime of a highly reliable product. We consider a special type of degradation test, namely the Step-Stress ADT, under which the stress is held constant and changed at some specified times. Even though many research efforts have been devoted to finding the optimum SSADT plan, most of the results are primarily empirical in nature. In this paper, we have derived the optimal SSADT plan based on the proposed Wiener model with a linear drift parameter–stress relationship in a formal manner. Our results suggest that the optimal SSADT used only the minimum and maximum values of stress for several commonly used optimization criteria. Furthermore, we derive the optimal allocation of inspections at each stress level.

A numerical example, a study of the step-stress accelerated degradation paths of the brightness of LED lamps, is provided for comparison. Both theoretical and simulation results suggest that the efficiencies could be improved by using the optimum simple SSADT plan. In addition, we observe that (through a numerical study), when the parameter-stress relationship is not a linear function, the optimal SSADT design needs to be investigated case by case due to the complexity of the information matrix and the optimal plan may not be a simple SSADT plan. Therefore, one promising and useful future research direction is to investigate the scenarios under which the analytical results in this paper continue to hold for other degradation models and parameter–stress relationships. Also, it will be useful to search for a robust SSADT plan in the sense that the plan could provide reasonable efficiencies to many objectives of interest simultaneously.

Acknowledgments

The authors would like to thank the Editor and two reviewers for their valuable comments and suggestions that have resulted in a significantly improved paper. Research of the first author began at Yuan Ze University and research of the second author was supported, in part, by a grant from the Providence University (No. PU101-11100-A07).

References

Balka, J., Desmond, A. F., & McNicholas, P. D. (2009). Review and implementation of curemodels based on first hitting times for Wiener processes. *Lifetime Data Analysis*, *15*, 147–176.

Bhattacharyya, G. K., & Fries, A. (1982). Fatigue failure models: Birnbaum– Saunders vs. inverse Gaussian. *IEEE Transactions on Reliability*, *31*, 439–441.

Boulanger, M., & Escobar, L. A. (1994). Experimental design for a class of accelerated degradation tests. *Technometrics*, *36*, 260–272.

Chhikara, R. S., & Folks, J. L. (1989). *The inverse Gaussian distribution: Theorem, methodology, and application*. New York: Marcel Dekker.

Chien, Y. H. (2008). A general age-replacement model with minimal repair under renewing free-replacement warranty. *European Journal of Operational Research*, *186*, 1046–1058.

Doksum, K. A., & Hoyland, A. (1992). Models for variable-stress accelerated life testing experiments based on Wiener processes and the inverse Gaussian distribution. *Technometrics*, *34*, 74–82.

Doksum, K. A., & Normand, S. L. T. (1995). Gaussian models for degradation processes-Part I: Methods for the analysis of biomarker data. *Lifetime Data Analysis*, *1*, 131–144.

Elsayed, E. A. (2012). Overview of reliability testing. *IEEE Transactions on Reliability*, *61*, 282–291.

Elwany, A., & Gebraeel, N. Z. (2009). Real-time estimation of mean remaining life using sensor-based degradation models. *Journal of Manufacturing Science and Engineering*, *131*, 051005-1–051005-9.

Ge, Z., Li, X., Jiang, T., & Huang, T. (2011). Optimal design for step-stress accelerated degradation testing based on D-optimality. *Proceedings of the Annual Reliability and Maintainability Symposium*, 1–6.

Gebraeel, N. Z., Lawley, M. A., Li, R., & Ryan, J. K. (2005). Residual-life distributions from component degradation signals: A Bayesian approach. *IIE Transactions*, *37*, 543–557.

Han, D., & Ng, H. K. T. (2013). Comparison between constant-stress and step-stress accelerated life tests under time constraint. *Naval Research Logistics*, *60*, 541–556.

Hu, C. H., Plante, R. D., & Tang, J. (2013). Statistical equivalency and optimality of simple step-stress accelerated test plans for the exponential distribution. *Naval Research Logistics*, *60*, 19–30.

Johnson, R. A., & Wichern, D. W. (2007). *Applied multivariate statistical analysis*. New Jersey: Pearson Prentice Hall.

Jung, G. M., & Park, D. H. (2003). Optimal maintenance policies during the post-warranty period. *Reliability Engineering and System Safety*, *82*, 173–185.

Lee, M. Y., & Tang, J. (2007). Modified EM-algorithm estimates of parameters of inverse Gaussian distribution based on time-censored Wiener degradation data. *Statistica Sinica*, *17*, 873–893.

Lee, M.Y., & Tang, J. (2013). On optimal design of accelerated degradation tests for Wiener degradation processes with time-censoring. Unpublished results.

Lehmann, A. (2009). Joint modeling of degradation and failure time data. *Journal of Statistical Planning and Inference*, *139*, 1693–1706.

Liao, H. T., & Elsayed, E. A. (2010). Equivalent accelerated life testing plans for log-location-scale distributions. *Naval Research Logistics*, *57*, 472–488.

Liao, C. M., & Tseng, S. T. (2006). Optimal design for step-stress accelerated degradation tests. *IEEE Transactions on Reliability*, *55*, 59–66.

Lim, H., & Yum, B. J. (2011). Optimal design of accelerated degradation tests based on Wiener process models. *Journal of Applied Statistics*, *38*, 309–325.

Lu, C. J., & Meeker, W. Q. (1993). Using degradation measures to estimate a time-to-failure distribution. *Technometrics*, *35*, 161–174.

Ma, H., & Meeker, W. Q. (2008). Optimum step-stress accelerated life test plans for log-location-scale distributions. *Naval Research Logistics*, *55*, 551–562.

Meeker, W. Q., & Escobar, L. A. (1998). *Statistical methods for reliability data*. New York: John Wiley & Sons.

Montgomery, D. (2013). *Design and analysis of experiments*. New York: John Wiley & Sons.

Murthy, M. N., & Sethi, V. K. (1965). Self-weighting design at tabulation stage. *Sankhya Series B*, *27*, 201–210.

Nalimov, V. V., Golikov, T. I., & Mikeshina, N. G. (1970). On practical use of the concept of D-optimality. *Technometrics*, *12*, 799–812.

Nelson, W. B. (2005). A bibliography of accelerated test plans. *IEEE Transactions on Reliability*, *54*, 194–197.

Nelson, W. B. (2005). A bibliography of accelerated test plans part II. *IEEE Transactions on Reliability*, *54*, 370–373.

Ng, H. K. T., Balakrishnan, N., & Chan, P. S. (2007). Optimal sample size allocation for tests with multiple levels of stress with extreme value regression. *Naval Research Logistics*, *54*, 237–249.

Padgett, W. J., & Tomlinson, M. A. (2004). Inference from accelerated degradation and failure data based on Gaussian process models. *Lifetime Data Analysis*, *10*, 191–206.

Pan, R., & Crispin, T. (2010). A hierarchical modeling approach to accelerated degradation testing data analysis: A case study. *Quality and Reliability Engineering International*, *27*, 229–237.

Park, C., & Padgett, W. J. (2005). Accelerated degradation models for failure based on geometric Brownian motion and gamma process. *Lifetime Data Analysis*, *11*, 511–527.

Park, J. I., & Yum, B. J. (1997). Optimal design of accelerated degradation tests for estimating mean lifetime at the use condition. *Engineering Optimization*, *28*, 199–230.

Seo, J. H., Jung, M., & Kim, C. M. (2009). Design of accelerated life test sampling plans with a nonconstant shape parameter. *European Journal of Operational Research*, *197*, 659–666.

Seshadri, V. (1999). *The inverse Gaussian distribution: Statistical theory and applications*. New York: Springer-Verlag.

Sherif, Y. S., & Smith, M. L. (1980). First-passage time distribution of Brownian motion as a reliability model. *IEEE Transactions on Reliability*, *29*, 425–426.

Sheu, S. H., & Chien, Y. H. (2005). Optimal burn-in time to minimize the cost for general repairable products sold under warranty. *European Journal of Operational Research*, *163*, 445–461.

Si, X. S., Wang, W., Chen, M. Y., Hu, C. H., & Zhou, D. H. (2013). A degradation path-dependent approach for remaining useful life estimation with an exact and closed-form solution. *European Journal of Operational Research*, *226*, 53–66.

Si, X. S., Wang, W., Hu, C. H., & Zhou, D. H. (2011). Remaining useful life estimation – A review on the statistical data driven approaches. *European Journal of Operational Research*, *213*, 1–14.

Tang, L. C., Yang, G. Y., & Xie, M. (2004). Planning of step-stress accelerated degradation test. *Proceedings of the Annual Reliability and Maintainability Symposium*, 287–292.

Tsai, C. C., Tseng, S. T., & Balakrishnan, N. (2011). Optimal burn-In policy for highly reliable products using Gamma degradation process. *IEEE Transactions on Reliability*, *60*, 234–245.

Tseng, S. T., Balakrishnan, N., & Tsai, C. C. (2009). Optimal step-stress accelerated degradation test plan for Gamma degradation processes. *IEEE Transactions on Reliability*, *58*, 611–618.
Tseng, S. T., & Peng, C. Y. (2004). Optimal burn-in policy by using integrated Wiener process. *IIE Transactions*, *36*, 1161–1170.
Tseng, S. T., Tang, J., & Ku, I. H. (2003). Determination of optimal burn-in parameters and residual life for highly reliable products. *Naval Research Logistics*, *50*, 1–14.
Tseng, S. T., & Wen, Z. C. (2000). Step-stress accelerated degradation analysis of highly reliable products. *Journal of Quality Technology*, *32*, 209–216.
Wang, X. (2010). Wiener process with random effects for degradation data. *Journal of Multivariate Analysis*, *101*, 340–351.
Wang, X., & Xu, D. (2010). An inverse Gaussian process model for degradation data. *Technometrics*, *52*, 188–197.
Whitmore, G. A., Crowder, M. J., & Lawless, J. F. (1998). Failure inference from a maker process based on a bivariate Wiener model. *Lifetime Data Analysis*, *4*, 229–251.
Whitmore, G. A., & Schenkelberg, F. (1997). Modeling an accelerated degradation data using Wiener diffusion with a time scale transformation. *Lifetime Data Analysis*, *3*, 27–45.
Wu, S. J., & Chang, C. T. (2002). Optimal design of degradation tests in presence of cost constraint. *Reliability Engineering and System Safety*, *76*, 109–115.
Ye, Z. S., Shen, Y., & Xie, M. (2012). Degradation-based burn-in with preventive maintenance. *European Journal of Operational Research*, *221*, 360–367.
Yu, H. F. (2003). Designing an accelerated degradation experiment by optimizing the estimation of the percentile. *Quality and Reliability Engineering International*, *19*, 197–214.
Yu, H. F. (2006). Designing an accelerated degradation experiment with a reciprocal Weibull degradation rate. *Journal of Statistical Planning and Inference*, *136*, 282–297.
Yu, H. F., & Tseng, S. T. (1998). On-line procedure for terminating an accelerated degradation test. *Statistica Sinica*, *8*, 207–220.
Yum, B. J., Lim, H., & Seo, S. K. (2007). Planning performance degradation tests – A review. *International Journal of Industrial Engineering Engineering-Theory Applications and Practice*, *14*, 372–381.
Zhang, J. R., Jiang, T. M., Li, X. Y., & Wang, L. Z. (2010). Optimization of step stress accelerated degradation test plans. *IEEE 17th International Conference on Industrial Engineering and Engineering Management*, 947–951.
Zhou, R., Gebraeel, N. Z., & Serban, N. (2012). Degradation modeling and monitoring of truncated degradation signals. *IIE Transactions*, *44*, 793–803.